Fourier-Motzkin Elimination Projects

- In 1827 Fourier published a method for solving linear inequalities in the real case. This method is known as Fourier-Motzkin elimination.
- The competition result has no impact at all on the grade.
- There is a Hall of Fame on the Course Home page which should be more populated by D-students I think.
- The first project is to make the implementation as fast as possible.
- The second project is to make the implementation as small as possible.

Projects

- You will be provided a file in which you will implement your algorithm and it contains code to run your algorithm, test it for correctness and measure the execution time, or actually the number of systems you can solve in 60 seconds.
- For the small size project, the size is measured as follows:
  1. `$ gcc name_fm.c -c` *plus all flags in the makefile*
  2. `$ size name_fm.o` which can print output similar to the following (for a different program)

```
idomeneo> size name_fm.o
   text    data     bss     dec     hex filename
   9060     168       9    9237    2415 name_fm.o
```

- The size to minimize is the sum of text, data, and bss, i.e. dec or hex for your file — but not the entire program as I had in mind earlier...
- Why is it good to write small code? Fewer cache misses and reduced memory requirements in a product save energy and product cost.
- BSS means block started by symbol and is data which will initialized to zeroes by the OS before the program starts — but it's a waste of

More Rules

- Each of the two projects will be ranked and the groups get a score with one point for the best project, two for the second best etc.
- The points for a group is added and this is the score for the group.
- In other words, if you focus on only one projects you will have no chance to win.
- If two groups get the same score, the group with the fastest code wins.

Fourier-Motzkin Elimination

- Assume we wish to determine if the following system of linear inequalities has a solution.

$$\begin{array}{rcrcr} 2x_1 & - & 11x_2 & \leq & 3 \\ -3x_1 & + & 2x_2 & \leq & -5 \\ x_1 & + & 3x_2 & \leq & 4 \\ -2x_1 & & & \leq & -3 \end{array} \tag{1}$$

- We will first eliminate x_2 from the system, and then check whether the remaining inequalities can be satisfied. To eliminate x_2, we start out with sorting the rows with respect to the coefficients of x_2:

$$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & \leq & -5 \\ x_1 & + & 3x_2 & \leq & 4 \\ 2x_1 & - & 11x_2 & \leq & 3 \\ -2x_1 & & & \leq & -3 \end{array} \tag{2}$$

Fourier-Motzkin Elimination

- First we want to have rows with positive coefficients of x_2, then negative, and lastly zero coefficients.
- Next we divide each row by the coefficient for x_2 if it is nonzero:

$$
\begin{array}{rcrcl}
\frac{-3}{2}x_1 & + & x_2 & \leq & \frac{-5}{2} \\
\frac{1}{3}x_1 & + & x_2 & \leq & \frac{4}{3} \\
\frac{-2}{11}x_1 & + & x_2 & \geq & \frac{-3}{11} \\
-2x_1 & & & \leq & -3
\end{array} \tag{3}
$$

Of course, the $\leq$ becomes $\geq$ when dividing with a negative coefficient. We can now rearrange the system to isolate x_2:

$$
\begin{array}{rcrcrcrcl}
 & & & & x_2 & \leq & \frac{3}{2}x_1 & - & \frac{5}{2} \\
 & & & & x_2 & \leq & \frac{-1}{3}x_1 & + & \frac{4}{3} \\
\frac{2}{11}x_1 & - & \frac{3}{11} & \leq & x_2 & & & & \\
-2x_1 & & & \leq & -3 & & & &
\end{array} \tag{4}
$$

- At this point, we make a record of the minimum and maximum values that x_2 can have, expressed as functions of x_1. We have:

$$b_2(x_1) \leq x_2 \leq B_2(x_1) \qquad (5)$$

where

$$\begin{aligned} b_2(x_1) &= \frac{2}{11}x_1 \\ B_2(x_1) &= \min(\tfrac{3}{2}x_1 - \tfrac{5}{2}, -\tfrac{1}{3}x_1 + \tfrac{4}{3}) \end{aligned} \qquad (6)$$

Fourier-Motzkin Elimination

- To eliminate x_2 from the system, we simply combine the inequalities which had positive coefficients of x_2 with those which had negative coefficients (ie, one with positive coefficient is combined with one with negative coefficient):

$$\begin{array}{lllll} \frac{2}{11}x_1 & - & \frac{3}{11} & \leq & \frac{3}{2}x_1 & - & \frac{5}{2} \\ \frac{2}{11}x_1 & - & \frac{3}{11} & \leq & -\frac{1}{3}x_1 & + & \frac{4}{3} \end{array} \tag{7}$$

- These are simplified and the inequality with the zero coefficient of x_2 is brought back:

$$\begin{array}{rcr} -\frac{29}{22}x_1 & \leq & -\frac{49}{22} \\ \frac{17}{33}x_1 & \leq & \frac{53}{33} \\ -2x_1 & \leq & -3 \end{array} \tag{8}$$

Fourier-Motzkin Elimination

- We can now repeat parts of the procedure above:

$$
\begin{aligned}
x_1 &\leq \frac{53}{17} \\
x_1 &\geq \frac{49}{29} \\
x_1 &\geq \frac{3}{2}
\end{aligned} \tag{9}
$$

- We find that

$$
\begin{aligned}
b_1() &= \max(49/29, 3/2) = 49/29 \\
B_1() &= 53/17
\end{aligned} \tag{10}
$$

  The solution to the system is $\frac{49}{29} \leq x_1 \leq \frac{53}{17}$ and $b_2(x_1) \leq B_2(x_1)$ for each value of x_1.

- In the projects you only have to return one if there is a solution and zero if there is none, but ***not*** the solutions!