Prov på delkurs Diskret Matematik

29 november 2002 kl 14–19 MA10A-C

Tillåtna hjälpmedel: Litteratur, egna anteckningar

Tentamen består av 3 uppgifter, varje uppgift kan ge 10 poäng. Deluppgifternas poäng anges inom [hakparenteser]. Observera att poänggivningen inte nödvändigtvis avspeglar deluppgiftens svårhetsgrad. Uppgifterna kan besvaras på svenska eller engelska.

Behandla högst en uppgift per papper (det går bra att behandla deluppgifter på samma papper). Skriv bara på ena sidan och markera varje sida med dina initialer. Skriv läsligt.

▶ a)[0] Hur många uppgifter får man behandla per papper?

▶ b)[0] Får man skriva på baksidan?

Det går bra att referera till kursboken av Rosen eller Husfeldts anteckningar. Referenser måste ha formen [R: *något*] eller [H: *något*], där *något* är ett avsnittsnummer, en kodbit, en räknad uppgift, etc. Skriv till exempel "genom att använda Lemma [R:7.4]".

Exercise 1 (Logic, Probability)

Given a sequence of Boolean variables $x_1, \ldots, x_n \in \{\mathrm{T}, \mathrm{F}\}$, a *random assignement* gives the value

$$x_i = \begin{cases} \mathrm{T}, & \text{with probability } p, \\ \mathrm{F}, & \text{with probability } 1-p. \end{cases}$$

For $p = \frac{1}{2}$ you can think of this as flipping a coin for each variable. If the coin comes up 'heads' then the variable is true, otherwise it's false. For example, for $p = \frac{1}{2}$ and $n = 5$ we can flip 5 coins like this:

$$\text{heads}, \text{heads}, \text{tails}, \text{heads}, \text{tails}, \tag{1}$$

which means that $x_1 = x_2 = x_4 = \mathrm{T}$ and $x_3 = x_5 = \mathrm{F}$. Under this random assignment the proposition $x_1 \vee x_5$ happens to be true, but $\neg x_1 \wedge (x_2 \vee x_5)$ is false.

Note. Except for the first two questions, you need to answer the following questions for general p ($0 \leq p \leq 1$) and general n ($n \in Z$, $n \geq 1$).

- ▶ a)[1] Find $p(x_1 \wedge x_2 \wedge x_4 = \mathrm{T})$ for $n = 5$ and $p = \frac{1}{2}$.
- ▶ b)[1] Find $p(x_1 \vee x_2 \vee \neg x_4 = \mathrm{T})$ for $n = 5$ and $p = \frac{1}{2}$.
- ▶ c)[1] Find $p(x_1 \wedge x_2 \wedge \cdots \wedge x_n = \mathrm{T})$, or, using other notation,

$$p\Big(\bigwedge_{i=1}^{n} x_i = \mathrm{T}\Big).$$

- ▶ d)[1] Find $p(x_1 \vee x_2 \vee \cdots \vee x_n)$, or, using other notation,

$$p\Big(\bigvee_{i=1}^{n} x_i = \mathrm{T}\Big).$$

- ▶ e)[1] Find $p\big((x_1 \vee x_2) \wedge \neg(x_3 \vee x_4) = \mathrm{T}\big)$.
- ▶ f)[1] (Harder) Find $p\big((x_1 \vee x_2) \wedge (x_1 \vee x_4) = \mathrm{T}\big)$. (Careful: the third variable is x_1, not x_3).
- ▶ g)[1] Find the probability that the conjunction of all variables is true given that their disjunction is true, in symbols,

$$p\Big(\bigwedge_{i=1}^{n} x_i = \mathrm{T} \;\Big|\; \bigvee_{i=1}^{n} x_i = \mathrm{T}\Big).$$

- ▶ h)[1] Find the probability that the disjunction of all variables is true given that their conjunction is true, in symbols,

$$p\Big(\bigvee_{i=1}^{n} x_i = \mathrm{T} \;\Big|\; \bigwedge_{i=1}^{n} x_i = \mathrm{T}\Big).$$

Let the stochastic variable N denote the number of true variables, formally

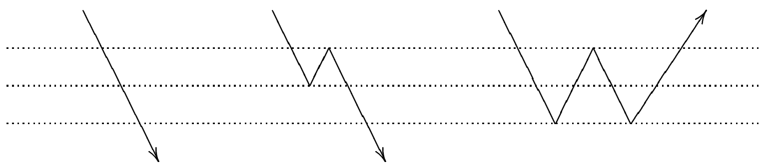
$$N = |\{\, i \mid x_i = \mathrm{T} \,\}|.$$

For example, if $x_1 = x_2 = x_4 = \mathrm{T}$ and $x_3 = x_5 = \mathrm{F}$ then $N = 3$.

- ▶ i)[2] Find $E(N)$.

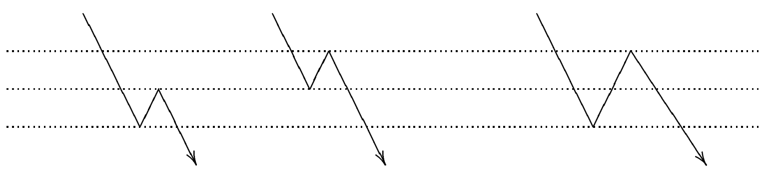
Exercise 2 (Combinatorics, induction)

Put two thick panes of glass (*glasskivor*) back-to-back. A ray of light (*ljusstråle*) can do two things at the boundaries: either it passes right through, or it is reflected. Below are three examples, showing rays that are reflected 0, 2, and 3 times, respectively. (The two glass panes are between the dotted lines, the ray of light is drawn as an arrow).



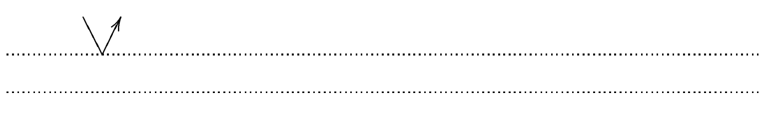
no reflections   two reflections   three reflections

Note that there are many different way in which a ray can be reflected, since it can 'bounce' at any of the three boundaries. For example, if we want exactly two reflections, then there are three different ways to do it:



Let B_n (the *bounce number*) denote the number of different ways a ray of light can be reflected exactly n times. We just saw that $B_2 = 3$.

We also saw in the first picture that $B_0 = 1$, since there is only one way to pass though the glass without getting reflected at all (namely, by passing right through it). And, to make life a little bit easier, let us agree that the ray actually needs to enter the glass panes, so the following does *not* count as a valid bounce:



► a)[1] Show that $B_1 = 2$ and $B_3 = 5$.

► b)[1] Find B_4.

► c)[4] Give a combinatorial argument for the formula

$$B_n = B_{n-1} + B_{n-2}.$$

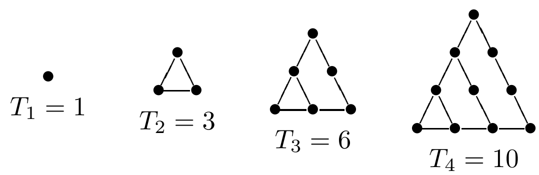
► d)[4] Prove by induction: For all $n \geq 0$ we have

$$B_0 + B_1 + B_2 + \cdots + B_n = B_{n+2} - 2.$$

Exercise 3 (Recurrences)

Ancient Greeks defined the *figurate numbers* to be the number of dots in certain geometrical configurations. We will look at *triangular* numbers and *pentagonal* numbers in this exercise.

The first few triangular numbers are 1, 3, 6, and 10:

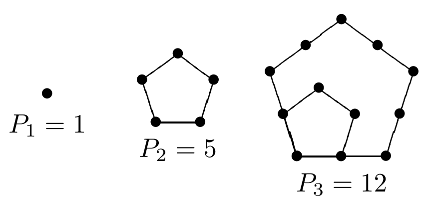


► a)[1] Find the 5th triangular number T_5.

► b)[4] Find a recurrence relation for T_n (remember to include the base case).

► c)[2] Solve the recurrence you found in the previous question. If you didn't solve the previous question, solve $D_1 = 4, D_n = D_{n-1} + 2n$ instead.

The pentagonal numbers are defined similarly, but with pentagons instead of triangles. It's also much harder to draw, so there are only three examples ($P_4 = 22$, but I won't draw it for you!).



► d)[3] Find and solve a recurrence relation for P_n.