# menu-proofwritingA Guide to Proof-Writing

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## 0102153menu-proofwriting25520451Introduction

Toward the end of Section 3.1, the text states that there is "no algorithm for proving theorems .... Such a procedure does not exist." This is true, but does not mean that proof-writing is purely an art, so that only those with exceptional talent and insight can possibly write proofs. Most proofs that students are asked to write in elementary courses fall into one of several categories, each calling for a systematic approach that can be demonstrated, imitated, and eventually mastered. We present some of these categories and techniques for working within them, organized as follows. This material supplements that found in the text and is intended to help get you started creating your own proofs. Also, studying the material in this Guide will help you understand better the proofs you read.

The material is organized as follows:

1. Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."
   - 1.1. Direct proofDirect proof
     - 1.1.1. Propositions having no hypothesisPropositions having no hypothesis
     - 1.1.2. Propositions having one or more hypothesesPropositions having one or more hypotheses
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     - 1.1.5. Proving equality of setsProving equality of sets
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     - 1.2.2. Proof by contradictionProof by contradiction
     - 1.2.3. Deriving conclusions of the form "$q$ or $r$"Deriving conclusions of the form "$q$ or $r$"

## 0102153menu-proofwriting255204511. Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."

Many defining properties in mathematics have the form $\forall x[P(x) \rightarrow Q(x)]$, representing the idea "All $P$'s are $Q$'s." (Cf. Examples 5, 20, and 21 in Section 1.3 of the text.) Some definitions involving this form are:

(i) A set $A$ is a *subset* of a set $B$: In symbols, $A \subseteq B$ if and only if $\forall x[(x \in A) \rightarrow (x \in B)]$. This is read in words, "$A$ is a subset of $B$ if and only if, *for every $x$, if $x \in A$, then $x \in B$.*" Less formally, $A$ is a subset of $B$ if and only if every element of $A$ is also an element of $B$. (Cf. Definition 3 in Section 1.4 of the text.)

(ii) A function $f$ is *one-to-one*: $f$ is one-to-one if and only if, *for every $x_1$* and $x_2$ in the domain of $f$, *if* $f(x_1) = f(x_2)$, *then* $x_1 = x_2$. (Cf. Definition 5 in Section 1.6 of the text.)

(iii) A relation $R$ on a set $A$ is *symmetric*: $R$ is symmetric if and only if, *for every $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.* (Cf. Definition 4 in Section 6.1 of the text.)

Many mathematical propositions that students are asked to prove have as their conclusion a statement involving a definition of the form just described. Some examples are:

(a) Prove that for all sets $A$ and $B$, $A \subseteq A \cup B$.

(b) Prove that for all sets $X$, $Y$, and $Z$, if $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$.

(c) Prove that for every function $f$ whose domain and codomain are subsets of the set of real numbers, if $f$ is strictly increasing (cf. Definition 6 in Section 1.6), then $f$ is one-to-one.

(d) Prove that for all relations $R_1$ and $R_2$ on a set $A$, if $R_1$ and $R_2$ are symmetric, then the relation $R_1 \cap R_2$ is symmetric.

Note that the *desired conclusion* in each of the propositions (a)–(d) is a statement involving one of the definitions (i)–(iii). Furthermore, propositions (b), (c), and (d) each have a *hypothesis*, a statement we are allowed to assume true and whose assumed truth, presumably, will play a role in deriving the conclusion. We begin our study of proof-writing methods by considering the very broad category known as *direct proof*.

### *0102153***1.1.** *0102153menu-proofwritingDirect proof*

An argument in which we prove a proposition in its originally-stated form is called a *direct proof*. Some forms of direct proof are discussed in Section 3.1 of the text. In the sections of this Guide that follow, we present various techniques for creating direct proofs. Attempting to write a direct proof of a proposition is usually our first line of attack. Direct proof contrasts with *indirect proof*, in which we prove a proposition by proving a different, but logically equivalent, form of the original proposition. We will introduce indirect proofs later, but will focus, in Examples 1–11, on various approaches to direct proofs.

#### *1.1.1. menu-proofwritingPropositions having no hypothesis*

Direct proofs of propositions like prop:a(a), having no hypothesis, tend to be simpler in their structure than the proofs that are required for propositions (b)–(d). Examples 1 and 2 demonstrate proofs for this simpler case.

**0102153menu-proof-examplesExample 1** Prove prop:aProposition (a): For all sets $A$ and $B$, $A \subseteq A \cup B$.

*0102153Solution*: The proof proceeds as follows: Let $A$ and $B$ be arbitrary sets. To prove $A \subseteq A \cup B$, let $x$ be an arbitrarily chosen element of $A$. [*Note*: We are *assuming* that $x \in A$.] We must prove that $x \in A \cup B$. By the definition of "union," this means we must prove that either $x \in A$ or $x \in B$. Since we know $x \in A$, by our assumption, the desired conclusion $x \in A$ or $x \in B$ follows immediately.

Let's dissect the proof in Example 1 and analyze what we did. Our starting point "... assume $x \in A$ ..." is an application of one of the most widely-used approaches to proof-writing, known as the *choose method*. The basic approach to deriving a conclusion of the form $\forall x[P(x) \to Q(x)]$, is to begin by choosing an arbitrary object (giving it a specific name such as "$x$") for which *it is assumed* that $P(x)$ is true. Our goal is to deduce that $Q(x)$ must consequently also be true. In Example 1, $P(x)$ is the assumption "$x \in A$" and $Q(x)$ is the desired conclusion "$x \in A \cup B$." We make the following additional observations:

- The object $x$ is a fixed, but arbitrarily chosen, element of the universe of discourse of $P(x)$ and $Q(x)$. We do not assign any specific value to $x$; rather we give the name "$x$" to a generic object [that is assumed to satisfy the propositional function $P(x)$] and use that name to keep track of the object as we proceed through the steps of the proof. The power of this approach is that any conclusion we draw about "this $x$" applies to every object $a$ for which the assumption $P(a)$ is true. This is valid by the rule of universal generalization; see Table 2 of Section 3.1.

- In the first part of a proof of a conclusion of the form $\forall x[P(x) \to Q(x)]$, called the "setting-up" of the proof, we choose $x$, assume $P(x)$ is true, and then write out what it would mean for $Q(x)$ to be true (in our example, "... to prove $x \in A \cup B$, we must prove that either $x \in A$ or $x \in B$"). Learning the process of setting up a proof in this category provides a fairly standardized, predictable, and almost mechanical beginning of a prospective direct proof. Furthermore, once we have written out these details, the remainder of the proof—the path from the assumption $P(x)$ to the desired conclusion $Q(x)$—is sometimes obvious.

- In the proof in Example 1, the path from what we assumed (i.e., $x \in A$) to the conclusion (i.e., $x \in A \cup B$) was obvious. In our proof, we stated that the conclusion "follows immediately." But was there something more than just "common sense" to justify that conclusion? Yes! The rule of inference $p \to p \vee q$ ("Law of Addition") is the underlying logical tool that justifies this step. This rule and other rules of inference are stated in the text, in Table 1, following Example 2 in Section 3.1. In the sample proofs that follow, we will make explicit reference to the rules of inference used (in an increasingly less obvious way as the proofs become more complex), even though it is common practice to apply these rules only implicitly, that is, without specific mention. To become proficient at writing proofs, you need to know how to use these rules of inference and when to use them.

**0102153menu-proof-examplesExample 2** Prove that for all sets $X$ and $Y$, $X \cap (Y \cup \overline{X}) \subseteq Y$.

*0102153Discussion.* Let $X$ and $Y$ be arbitrary sets. To prove $X \cap (Y \cup \overline{X}) \subseteq Y$, let $a$ be an arbitrarily chosen element of $X \cap (Y \cup \overline{X})$. [We could also say "Assume $a \in X \cap (Y \cup \overline{X})$."] We must prove $a \in Y$. [This concludes the setting-up of the proof. Now we must figure out how to get from the assumption to the conclusion. To do that, we begin by analyzing what our assumption means.]

Since $a \in X \cap (Y \cup \overline{X})$, we know that $a \in X$ and $a \in Y \cup \overline{X}$. The latter, in turn, tells us that either $a \in Y$ or $a \in \overline{X}$, that is, either $a \in Y$ or $a \notin X$. [Note that the preceding sentence makes the first mention of the desired conclusion $a \in Y$.] Now, can we infer the conclusion $a \in Y$ from the known "either $a \in Y$ or $a \notin X$"? This would require a rule of inference "$(p \vee q) \to p$" (the converse of the Law of Addition). This is not a valid inference since $(p \vee q) \to p$ is not a tautology, so this approach does not work. Note, however, that not only do we know "either $a \in Y$ or $a \notin X$" from our assumption, but we also know that $a \in X$. Thus, what we know from our assumption has the form $(p \vee q) \wedge \neg q$. [*Note*: $p$ is "$a \in Y$" and $q$ is "$a \notin X$," so $\neg q$ is "$a \in X$."] Does Table 1 in Section 3.1 give us a conclusion that follows from this premise? It does! The Law of

Disjunctive Syllogism, $[(p \vee q) \wedge \neg q] \to p$, enables us to draw the conclusion $p$, that is, $a \in Y$, the desired conclusion. [*Note*: As stated in Table 1, the roles of $p$ and $q$ are reversed from what we have here, but that is of no consequence.]

Before moving on, we rewrite the preceding proof, leaving out explanatory comments. What remains provides a representative view of what a typical proof looks like:

"Let $X$ and $Y$ be arbitrary sets. To prove $X \cap (Y \cup \overline{X}) \subseteq Y$, assume $a \in X \cap (Y \cup \overline{X})$. We must prove $a \in Y$. By our assumption, we know $a \in X$ and $a \in Y \cup \overline{X}$; therefore $a \in X$ and either $a \in Y$ or $a \in \overline{X}$. Thus we know that either $a \in Y$ or $a \notin X$; but we also know that $a \in X$, so $a \notin X$ is false. Hence we conclude $a \in Y$, as desired."

At this point, you may wish to try some relevant exercises in the text, such as Exercises 10(a,c) and 12(a,b,c) in Section 1.5. You should find the principles from ex1Examples 1 and ex22 helpful in attempting these exercises.

*1.1.2. menu-proofwritingPropositions having one or more hypotheses*

As we work through the steps of a prospective proof, the tools at our disposal in moving toward a desired conclusion are

- the assumption(s) we are entitled to make at the outset in setting up the proof,
- assumed axioms and previously-proved theorems (if any), and
- rules of inference from logic, such as $p \to (p \vee q)$, used in ex1Example 1, and $[(p \vee q) \wedge \neg q] \to p$, used in ex2Example 2. (See Table 1 in Section 3.1 for additional rules of this type.)

In addition to these, most propositions we are asked to prove contain

- one or more *hypotheses*, statements whose truth is to be assumed in the proof and which, we expect, will be used as part of the argument leading to the conclusion.

Example 3 provides our first instance of a proposition in which a hypothesis is to be assumed in deriving a conclusion of the form $\forall x[P(x) \to Q(x)]$.

**0102153menu-proof-examplesExample 3** Prove prop:bProposition (b): For all sets $X$, $Y$, and $Z$, *if* $X \subseteq Y$, *then* $X \cap Z \subseteq Y \cap Z$.

*0102153Proof:* Let $X$, $Y$, and $Z$ be sets such that $X \subseteq Y$. To prove $X \cap Z \subseteq Y \cap Z$, assume $b \in X \cap Z$. To prove $b \in Y \cap Z$, we must prove $b \in Y$ and $b \in Z$. [This marks the end of "setting up the proof." Now we must return to our assumption and the hypothesis, and begin to analyze what they mean and what information we can draw from them.] By our assumption, we know that

$b \in X$ and $b \in Z$, so, in particular, $b \in Z$, one of our two desired conclusions. Furthermore, since $b \in X$ (part of our assumption) and since $X \subseteq Y$ [here we are, for the first time, bringing in the hypothesis], we may conclude that $b \in Y$, our other desired conclusion.

The following feature of the proof in Example 3 is very important. In setting up the argument at the outset, we applied the choose method to the desired conclusion, not the hypothesis. Thus our initial statement was "... assume $b \in X \cap Z$." A common mistake by beginning students is to begin with "... assume $b \in X$ ...," erroneously focusing at the start of the proof on the hypothesis rather than on the desired conclusion. Note that we did not employ the hypothesis until the very end of the proof!

In the last sentence of the proof in Example 3, we concluded $b \in Y$ from knowing $b \in X$ and $X \subseteq Y$. Let us consider why this conclusion is justified. The truth of $X \subseteq Y$ means that the proposition $\forall x[(x \in X) \to (x \in Y)]$ is true. Thus, in particular, the proposition $(b \in X) \to (b \in Y)$ is true, where $b$ is the specific object we are working with in the proof. Since $b \in X$ is true and the "if...then" statement $(b \in X) \to (b \in Y)$ is true, the truth of $b \in Y$ follows from the rule of inference modus ponens (cf. Table 1 in Section 3.1 of the text). Note, once again, that a rule of inference has played an important, though implicit, role in a proof!

The principles discussed thus far apply to every proof of a proposition whose conclusion has the logical form $\forall x[P(x) \to Q(x)]$, and not just to proofs that one set is a subset of another. Examples 4 and 5 illustrate this.

**0102153menu-proof-examplesExample 4** Prove that every nonconstant linear function $f(x) = Mx + B$, $M \neq 0$, is one-to-one.

*0102153Proof:* Let $M$ be a nonzero real number. Let $x_1$ and $x_2$ be real numbers and assume that $f(x_1) = f(x_2)$. We must prove that $x_1 = x_2$. Since $f(x_1) = Mx_1 + B$ and $f(x_2) = Mx_2 + B$, we have $Mx_1 + B = Mx_2 + B$. By a rule of elementary algebra, if $Mx_1 + B = Mx_2 + B$, then $Mx_1 = Mx_2$. Since $Mx_1 = Mx_2$ and $M \neq 0$, by hypothesis, we conclude by another rule of elementary algebra that $x_1 = x_2$, as desired.

**0102153menu-proof-examplesExample 5** Prove prop:dProposition (d): For all relations $R_1$ and $R_2$ on a set $A$, if $R_1$ and $R_2$ are symmetric, then the relation $R_1 \cap R_2$ is symmetric.

*0102153Proof:* Let $A$ be an arbitrary set and let $R_1$ and $R_2$ be symmetric relations on $A$. To prove that the relation $R_1 \cap R_2$ is symmetric, let $x$ and $y$ be arbitrary elements of $A$ and assume that $(x, y) \in R_1 \cap R_2$. We must prove $(y, x) \in R_1 \cap R_2$, that is, $(y, x) \in R_1$ and $(y, x) \in R_2$. [End of set-up!] Now since

$(x, y) \in R_1 \cap R_2$ (by assumption), we know that $(x, y) \in R_1$ and $(x, y) \in R_2$. Since $(x, y) \in R_1$ and $R_1$ is symmetric (by hypothesis), $(y, x) \in R_1$. This is one of our desired conclusions. Since $(x, y) \in R_2$ and $R_2$ is symmetric (by hypothesis), $(y, x) \in R_2$, the second of our two desired conclusions. With this, the proposition is proved.

At this point, you may want to practice applying the principles from Examples 3–5 in the following exercises:

- Prove that for all sets $A$ and $B$, if $A \cap B = A$, then $A \subseteq B$.
- Prove that for all sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (This is Exercise 9 in Section 1.4 of the text.)
- Prove that the function $g(x) = x^3$ is one-to-one.

*1.1.3. menu-proofwritingDisproving false propositions having conclusions of the form* $\forall x[P(x) \to Q(x)]$

Sometimes we are faced with a proposition that we must either "prove or disprove." We are not told in advance whether the proposition is true. If it is false, then it will of course be impossible to write a correct proof of the proposition. Time spent trying to do so may provide insight, but cannot ultimately lead to a valid proof. Example 6 illustrates how we should approach this type of problem.

**0102153menu-proof-examplesExample 6** Prove or disprove the converse of prop:bProposition (b): For all sets $X$, $Y$, and $Z$, *if* $X \cap Z \subseteq Y \cap Z$, *then* $X \subseteq Y$. (Cf. ex3Example 3.)

*0102153Discussion.* Suppose we try first to approach this proposition in the manner of previous examples. Our setting up of a "proof" would read as follows: "Let $X$, $Y$, and $Z$ be arbitrary sets such that $X \cap Z \subseteq Y \cap Z$. To prove $X \subseteq Y$, let $w \in X$; we must prove $w \in Y$." At this point, we must return to the hypothesis $X \cap Z \subseteq Y \cap Z$ and ask whether, in combination with the assumption $w \in X$, it leads to the conclusion $w \in Y$. If we could get $w$ to lie in $X \cap Z$, then we could invoke the hypothesis to conclude $w \in Y \cap Z$, which would imply $w \in Y$, the desired conclusion. However, we know from our assumption only that $w \in X$; in order to conclude $w \in X \cap Z$, we would need to know that $w \in Z$, which we do not!

With this, our attempt to write a direct proof breaks down, leaving us with two possibilities. Either there is another route to a proof, or perhaps, the proposition we are trying to prove is false. If we do not know whether a general proposition is true or false, and our initial attempts at a proof fail, we should do some experimenting to see whether we can find a *counterexample*, i.e., a specific example that contradicts the truth of the proposition. Before we can do that, we must formulate precisely the negation of the proposition. Logically the negation of a proposition "for every $x$, if $P(x)$, then $Q(x)$" is "there exists $x$ such that $P(x)$ but not $Q(x)$." In symbols, $\neg\forall x[P(x) \to Q(x)]$ is equivalent to $\exists x[P(x) \wedge \neg Q(x)]$. (Cf. Example 25 in Section 3.1 of the text.) In this example, the negation is "there exist sets $X$, $Y$, and $Z$ such that $X \cap Z \subseteq Y \cap Z$, but $X$ is not a subset of $Y$." Can we find specific sets $X$, $Y$, and $Z$ satisfying this statement? Consider the sets $X = \{4, 7, 8, 11\}$, $Y = \{2, 7, 8, 9, 11\}$, and $Z = \{1, 7, 8, 9, 10, 11, 12\}$. Note that $X \cap Z = \{7, 8, 11\}$ and $Y \cap Z = \{7, 8, 9, 11\}$, so $X \cap Z \subseteq Y \cap Z$. However $X$ is clearly not a subset of $Y$. Hence we have a counterexample; the proposition in question is false!

Note that a single counterexample to a general proposition is sufficient to prove that proposition false. This is a far cry from what is required to prove a general proposition true, when the domain of discourse is infinite. Since we can never exhaust all the possible examples, no number of specific cases that affirm a proposition are sufficient to establish its truth in general. We must write a general proof in order to do that—the process of writing such proofs is the major topic you are now studying, and one to which we will return shortly.

In the last sentence of the Discussion of Example 6, we stated that "$X$ is clearly not a subset of $Y$." How do we justify this statement formally? Recall that "$X \subseteq Y$" is defined by "$\forall w[(w \in X) \to (w \in Y)]$." Hence the proposition "$\exists w[(w \in X) \wedge (w \notin Y)]$" corresponds to "$X$ is not a subset of $Y$." For the sets $X = \{4, 7, 8, 11\}$ and $Y = \{2, 7, 8, 9, 11\}$, we note that, choosing $w = 4$, we have $4 \in X$, but $4 \notin Y$. This particular example is all that is needed to conclude that $X$ is not a subset of $Y$. (Note incidentally that the choice of $w$ used to prove that $X$ is not a subset of $Y$, $w = 4$, has the property that $w \notin Z$. This is not surprising since, in our initial attempt to prove the false proposition in Example 6, the obstacle we could not overcome was that our arbitrarily chosen $w$ did not need to lie in $Z$.)

The following exercises are germane to the issues raised by Example 6 and the paragraphs following it.

- Prove or disprove: For all sets $X$, $Y$, and $Z$, $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap Z$.
- Prove or disprove: For all sets $X$, $Y$, and $Z$, if $X \subseteq Z$, then $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap Z$.

*1.1.4. menu-proofwritingThe tactic of division into cases*

As propositions we are asked to prove become more complex, we must expand our arsenal of tools that are effective for proceeding toward the desired conclusion of a proposition, once we have finished setting up the argument. The applications of the choose method that occur in Examples 7 and 8, which follow, demonstrate a new such tool, known as *division into cases*, which is useful in some proofs.

**0102153menu-proof-examplesExample 7** Prove that for all sets $A$ and $B$, $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$.

*0102153Proof:* Let $A$ and $B$ be arbitrary sets. To prove $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$, assume $x \in (A \cap B) \cup (A \cap \overline{B})$. We must prove $x \in A$. By our assumption, we know that either $x \in A \cap B$ or $x \in A \cap \overline{B}$, that is, either $x \in A$ and $x \in B$, or else $x \in A$ and $x \in \overline{B}$. [*Note*: We don't know which of these two is the case, but we do know that at least one of them must be true.] Hence, at this point, we divide the argument into two exhaustive cases:

*Case I* Suppose that $x \in A$ and $x \in B$. Then, in particular, $x \in A$ [by the rule of inference $(p \wedge q) \to p$], so the desired conclusion obtains in this case.

*Case II* Suppose that $x \in A$ and $x \in \overline{B}$. Then, again, $x \in A$, so the desired conclusion is again verified.

Under either of the only two possible cases, we have $x \in A$, the desired conclusion.

**0102153menu-proof-examplesExample 8** Prove that for all sets $A$ and $B$, $A \subseteq (A \cap B) \cup (A \cap \overline{B})$.

*0102153Proof:* Let $A$ and $B$ be arbitrary sets. To prove $A \subseteq (A \cap B) \cup (A \cap \overline{B})$, assume $x \in A$. We must prove $x \in (A \cap B) \cup (A \cap \overline{B})$. To do this, we must prove that either $x \in A \cap B$ or $x \in A \cap \overline{B}$, that is, either $x \in A$ and $x \in B$, or else $x \in A$ and $x \in \overline{B}$. [Recall that our assumption is that $x \in A$. This assumption involves only the set $A$. The problem we must solve is how to bring the relationship between $x$ and the set $B$ into the discussion.] We note that, necessarily, either $x \in B$ or $x \notin B$, by the tautology $p \vee \neg p$. Having noted this, we consider two cases:

*Case I* Suppose that $x \in B$. Then, since $x \in A$ [by our assumption], we have $x \in A$ and $x \in B$, one of the two alternatives in our desired conclusion.

*Case II* Suppose that $x \notin B$. Equivalently, $x \in \overline{B}$. Then, since $x \in A$, we have $x \in A$ and $x \in \overline{B}$, the other of the two alternatives in our desired conclusion.

Compare the proofs in Examples 7 and 8, both involving division into cases. The second proof illustrates somewhat more creativity than does the first. It requires a slightly more active role on our part, involving the tautology $p \vee \neg p$. The truth of this tautology is "common sense." Once the idea of bringing this division into cases into the argument is suggested, virtually anyone would agree that it is correct and, furthermore, is an effective step at this stage. The difficult part for you, as a beginning student just learning to write proofs, is thinking of this idea on your own.

Many complex proofs require that some creative idea be brought in from outside the basic structure (i.e., the setting up) of the argument. This is the aspect of proof-writing that is not mechanical. It is learned from experience and by an active interest in the "why" in mathematics. It is fostered by developing the habit of having firmly in mind all the statements, pertaining to the problem at hand, that we know to be true, and by being willing to try to apply these statements until we find one that works.

Here are some exercises that involve the choose method and division into cases:

- Prove that for all sets $X$, $Y$, and $Z$, if $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.
- Prove that for all sets $A$, $B$, and $C$, if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.
- Prove that for all sets $X$, $Y$, and $Z$, if $X \cap Z \subseteq Y \cap Z$ and $X \cap \overline{Z} \subseteq Y \cap \overline{Z}$, then $X \subseteq Y$.

*1.1.5. menu-proofwritingProving equality of sets*

Three approaches to proving equality of sets are discussed in Examples 10–12 in Section 1.5 of the text. Of these, the first, known as *mutual inclusion* (introduced in Example 10), is the most generally applicable. In addition, it expands naturally on the earlier material in this Guide, so we give that approach additional emphasis here.

A formal version of the definition of equality of sets given in the text (cf. Definition 2 in Section 1.4) is

$$A = B \quad \text{if and only if} \quad \forall x[(x \in A) \leftrightarrow (x \in B)].$$

The latter proposition is equivalent (cf. Example 21 in Section 3.1 of the text) to

$$\forall x\{[(x \in A) \rightarrow (x \in B)] \wedge [(x \in B) \rightarrow (x \in A)]\},$$

which, in turn, is equivalent to

$$\{\forall x[(x \in A) \rightarrow (x \in B)]\} \;\wedge\; \{\forall x[(x \in B) \rightarrow (x \in A)]\},$$

which is the definition of

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

We may prove that two sets are equal by proving that each is a subset of the other. We illustrate this approach in Examples 9 and 10.

**0102153menu-proof-examplesExample 9** Prove that for all sets $A$ and $B$, $(A \cap B) \cup (A \cap \overline{B}) = A$.

*0102153Proof:* We may prove the desired equality by proving mutual inclusion, i.e., that each of the two sets is a subset of the other. The inclusion $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$, however, is precisely what we proved already in ex7Example 7.

The other inclusion $A \subseteq (A \cap B) \cup (A \cap \overline{B})$ was proved in ex8Example 8. Having written these two proofs, we have established the desired equality.

In Example 10, we encounter a conclusion of equality that is preceded by a hypothesis.

**0102153menu-proof-examplesExample 10** Prove that for all sets $A$ and $B$, if $A \subseteq B$, then$A \cup B = B$.

*0102153Proof:* Let $A$ and $B$ be arbitrary sets such that $A \subseteq B$. We may prove $A \cup B = B$ by proving $B \subseteq A \cup B$ and $A \cup B \subseteq B$.

1. $B \subseteq A \cup B$ is essentially prop:aProposition (a), proved earlier in ex1Example 1. [Note that the hypothesis of the theorem is not required to establish this inclusion. Like Example 17 in Section 3.1 of the text, the proof in this direction is *trivial*.]

2. To prove $A \cup B \subseteq B$, given the hypothesis $A \subseteq B$, we proceed by the choose method. Assume $x \in A \cup B$. We must prove that$x \in B$. By our assumption, we know that either $x \in A$ or $x \in B$. Since we do not know which of these two statements is true, we divide the argument into cases:

*Case I* Suppose that $x \in A$. Then, since $A \subseteq B$, by hypothesis, we have $x \in B$, as desired. [Recall the middle paragraph of the discussion between Examples 3 and 4, on the role of modus ponens.]

*Case II* Suppose that $x \in B$. Since this is the desired conclusion, the result is trivially true in this case.

We conclude $x \in B$, as desired.

In some circumstances, it is possible to prove set equality using a single chain of valid equations, thus avoiding the sometimes cumbersome mutual-inclusion approach. One such circumstance is in proving any of the set identities in Table 1 in Section 1.5 of the text. Each of these results is a set-theory version of a corresponding equivalence of propositions in logic, covered in Indirect proofSection 1.2. Each can be proved using the approach of Example 11 in Section 1.5 of the text.

Another approach to proving an equality of sets is to use other equalities proved previously. Suppose the following identities of set theory have already been proved:

$$\text{For all sets } A,\ B,\ \text{and } C,\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \qquad [1]$$

$$\text{For every set } B,\ B \cup \overline{B} = U \quad (U \text{ represents the universal set.}) \qquad [2]$$

$$\text{For every set } C,\ C \cap U = C. \qquad [3]$$

On these bases, we can give a proof of the proposition proved in ex9Example 9 that does not use mutual inclusion.

**0102153menu-proof-examplesExample 11** Prove that for all sets $A$ and $B$, $(A \cap B) \cup (A \cap \overline{B}) = A$.

*0102153Proof:* Let $A$ and $B$ be arbitrary sets. Then we have

$$(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) \qquad \text{(by [1])} = A \cap U \qquad \text{(by [2])} = A,$$

as desired.

Note that, in applying the result [1], we used the special case $C = \overline{B}$ of the identity [1] (which is the property of *distributivity* of intersection over union—see Table 1 in Section 1.5 of the text). The technique of using a special case of a known result is called *specialization*. Whenever you find yourself saying "in particular," in making an inference from a known general fact in the course of an argument, you are using the specialization tactic. Like division into cases, specialization is a sometimes-useful technique for proceeding beyond the initial setting up of a proof toward the desired conclusion. Formally, it is justified by the rule of universal instantiation, shown in Table 2 of Section 3.1.

### *01021531.2. 0102153menu-proofwritingIndirect proof*

Sometimes it is convenient, or even necessary, to prove a form of a proposition that is different from the original, but logically equivalent to it. Whenever we write a proof in such a form, we are writing an indirect proof . Three common forms of indirect proof are based on three logical equivalences of pairs of propositions:

- $(\neg q) \to (\neg p)$ is equivalent to $p \to q$ [4]
- $(\neg p) \to (q \wedge \neg q)$ is equivalent to $p$ [5]
- $(p \wedge \neg q) \to r$ is equivalent to $p \to (q \vee r)$. [6]

The equivalence [4] is the basis for the form of an indirect proof known as *proof by contrapositive*. The equivalence [5] justifies the form of an indirect proof called *proof by contradiction*. The equivalence [6] underlies a standard approach to deriving a conclusion involving alternatives, i.e., having the form "either $q$ or $r$."

*1.2.1. menu-proofwritingProof by contrapositive*

Sometimes it is difficult to see how to prove a proposition of the form $\forall x[P(x) \to Q(x)]$ by starting with the assumption that $P(x)$ is true. (Recall Section 1.1.2 of this Guide.) What can you do if you cannot see how to deduce the conclusion $Q(x)$ from the assumption $P(x)$ and any additional given hypotheses (if there are any)? Sometimes, in such cases, assuming the negation $\neg Q(x)$ of the conclusion provides a better match with known facts or the other given hypotheses, and the two together lead readily to the negation $\neg P(x)$ of the original assumption. An argument in this form is an instance of proof by contrapositive. See Example 15 in Section 3.1 of the text for an example of such a proof. Our Example 12 provides another illustration of the method.

**0102153menu-proof-examplesExample 12** Prove prop:cProposition (c): For every function $f$ whose domain and codomain are subsets of the set of real numbers, if $f$ is strictly increasing, then $f$ is one-to-one.

*0102153Discussion.* Let $f$ be any function that is strictly increasing. To show that $f$ is one-to-one using the original form of the definition, we would let $x_1$ and $x_2$ be real numbers in the domain of $f$ and assume $f(x_1) = f(x_2)$. We would then have to prove $x_1 = x_2$. We have completed the setting up of a direct proof, but have no way of using the hypothesis that $f$ is strictly increasing [i.e., "... if $x_1 < x_2$, then $f(x_1) < f(x_2)$"]. The assumption $f(x_1) = f(x_2)$ simply does not "match up" with the "if part" of the hypothesis in a way that permits us to proceed anywhere from that hypothesis.

However, suppose we decide instead to derive the contrapositive of the definition of one-to-one. Under this approach, we will begin by assuming that $x_1 \neq x_2$. Our goal will then be to prove that $f(x_1) \neq f(x_2)$. We proceed from here as follows. Since $x_1 \neq x_2$ (by assumption), then it must be that either $x_1 < x_2$ or $x_2 < x_1$. We consider two cases:

*Case I* If $x_1 < x_2$, then since $f$ is strictly increasing, we may conclude $f(x_1) < f(x_2)$, so, in particular, $f(x_1) \neq f(x_2)$, as desired.

*Case II* If $x_2 < x_1$, then since $f$ is strictly increasing, we may conclude $f(x_2) < f(x_1)$, so $f(x_1) \neq f(x_2)$, again as desired.

Another situation in which a proof by contrapositive is sometimes called for is one in which the conclusion of the proposition has a much simpler logical form than does the hypothesis. We illustrate this in the following example.

**0102153menu-proof-examplesExample 13** Suppose that $a$ is a real number satisfying the property $\forall M{>}0\,(|a| < M)$. Then $a = 0$.

*0102153Discussion.* Note the simple form of the conclusion. Rather than involving a definition having an "if...then" form, as in most of our earlier examples, it is simply the flat statement that $a = 0$. If we try to begin a direct proof by focusing on the desired conclusion, then there is really no place to begin, no basis for making the kind of assumption that is needed to get the proof "off the ground."

So instead we proceed by contraposition. Our approach will be to assume $a \neq 0$ and try to deduce the negation of the hypothesis. This negation may be formulated (cf. Table 1 in Section 1.3 of the text)

$$\exists M{>}0\,(|a| \geq M). \qquad [7]$$

We need only produce a positive $M$ whose value does not exceed that of the absolute value of the nonzero $a$. We take $M = |a|$, noting that this value of $M$ clearly satisfies the condition [7].

Here is a third circumstance in which a proof by contrapositive is appropriate. Suppose a proposition of the form "if $p$ and $q$, then $r$" is known to be true, and we are asked to prove that $p$ and the negation of $r$ together imply the negation of $q$. We may always proceed, using contraposition, by assuming that the negation of $q$ is false, that is, that $q$ is true. Then since $p$ is true, by hypothesis, we have that $p$ and $q$ are both true, so, by the known proposition, we may conclude that $r$ is true, contradicting the fact that $\neg r$ is one of the hypotheses. You will have an opportunity to apply this approach in the third and fourth of the exercises that follow.

- Prove that for all sets $A$ and $B$, if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.
- Prove that if a linear function $f(x) = Mx + B$ is one-to-one, then $M \neq 0$.
- Suppose it is known that "every sum or difference of two integers is an integer." Use this result to prove that for all real numbers $x$ and $y$, if $x$ is an integer and $x + y$ is an integer, then $y$ is an integer. Prove also that the sum of an integer and a noninteger must be a noninteger.
- Prove that for all sets $A$ and $B$ and for every object $x$, if $x \in A$ and $x \notin A \cap B$, then $x \notin B$.

*1.2.2. menu-proofwritingProof by contradiction*

The idea behind the equivalence Indirect proof[5] is that we may prove a conclusion $p$ by showing that the denial of $p$ leads to a contradiction. Actually, proof by contrapositive is a form of proof by contradiction. For if, in a proof that $p$ implies $q$, we assume the truth of $p$ (as we are entitled to do) and then use the negation of $q$ to derive $\neg p$, then we have obtained the contradiction $p \wedge \neg p$. Another circumstance in which proof by contradiction is the standard approach is any proof of a theorem in set theory in which the conclusion asserts that some set equals the empty set.

**0102153menu-proof-examplesExample 14** Prove that for all sets $A$ and $B$, if $A \subseteq B$, then $A \cap \overline{B} = \emptyset$.

*0102153Discussion.* A direct approach would be to establish the equality $A \cap \overline{B} = \emptyset$ using mutual inclusion. Indeed the containment in one direction, $\emptyset \subseteq A \cap \overline{B}$, is true automatically, based on the principle that the empty set is a subset of every set (this result is vacuously true—recall Example 16 in Section 3.1 of the text). However, for the containment in the other direction, $A \cap \overline{B} \subseteq \emptyset$, the approach "assume $x \in A \cap \overline{B} \ldots$ we must prove $x \in \emptyset$" is doomed to failure, since the conclusion "$x \in \emptyset$" can never be reached.

Since we are unable to write a direct proof, we proceed by contradiction. Let $A$ and $B$ be arbitrary sets such that $A \subseteq B$. Assume that $A \cap \overline{B} \neq \emptyset$. Then there exists some object that lies in $A \cap \overline{B}$; let us call it $c$. Since $c \in A \cap \overline{B}$, we know that $c \in A$ and $c \in \overline{B}$. Since $c \in A$ and $A \subseteq B$, by hypothesis, we have $c \in B$. Thus we have $c \in B$ and $c \in \overline{B}$, so $c \in B$ and $c \notin B$. This is a contradiction of the form $p \wedge \neg p$, so our proof is complete .

A classic example of a proof by contradiction is provided in the text in Example 18 of Section 3.1, which shows that $\sqrt{2}$ is irrational. Here are some exercises:

- Prove that for every set $A$, $A \cap \overline{A} = \emptyset$.
- Prove that for all sets $A$ and $B$, if $(B \cap \overline{A}) \cup (\overline{B} \cap A) = B$, then $A = \emptyset$.
- Prove that for all sets $A$ and $B$, $(\overline{A} \cup \overline{B}) \cap (\overline{A} \cup B) \cap (A \cup \overline{B}) \cap (A \cup B) = \emptyset$.

### *1.2.3. menu-proofwritingDeriving conclusions of the form "q or r"*

The equivalence Indirect proof[6] becomes relevant to the writing of proofs when we must derive a conclusion involving alternatives, $q$ or $r$. For a proposition of this type, there may be no circumstance under which we can be sure which of the alternatives is true, only that at least one of them must be true under every circumstance in which the hypothesis is true. Because of this, we are unable to determine whether to set up a direct proof based on the conclusion $q$ or on the conclusion $r$. (Indeed, usually no such proof is possible.) Fortunately there is an indirect approach, based on the equivalence [6], that enables us to get around this difficulty. Rather than attempting to prove directly that $q$ or $r$ follows from $p$, we may replace this problem by the problem of showing that $r$ follows from $p$ and $\neg q$ (or else that $q$ follows from $p$ and $\neg r$—either approach will do the job).

A classic example of a proof in this category is the following theorem from elementary algebra: "For all real numbers $x$ and $y$, if $xy = 0$, then $x = 0$ or $y = 0$." Clearly we should set up this proof by letting $x$ and $y$ be real numbers such that $xy = 0$. But at this point there is no evident way of proceeding toward the conclusion that one or the other of $x$ and $y$ (we know not which) equals 0. The escape is to make the additional assumption that $x \neq 0$, with the goal of proving that therefore $y$ must equal zero. Since $x \neq 0$, its reciprocal $1/x$ must exist, and we may write the chain of equations $y = 1 \cdot y = [(1/x)(x)](y) =$

$(1/x)(xy) = (1/x)(0) = 0$, so $y = 0$, as desired. [Note that this chain of equations also uses the facts that multiplication of real numbers is associative and that the product of every real number with zero equals zero.]

A problem in set theory in which this approach is sometimes useful is proving that one set is a subset of the union of two other sets. This is demonstrated in Example 15.

**0102153menu-proof-examplesExample 15** Prove that for all sets $A$ and $B$, $A \subseteq B \cup (A \cap \overline{B})$.

*0102153Proof:* Let $A$ and $B$ be arbitrary sets. To prove $A \subseteq B \cup (A \cap \overline{B})$, assume that $x \in A$. We must prove that $x \in B \cup (A \cap \overline{B})$, that is, either $x \in B$ or $x \in A \cap \overline{B}$. Since our desired conclusion is now seen to have the form "either $q$ or $r$," we take the approach suggested by the equivalence [6], and assume that $x \notin B$. Our goal now becomes to prove that, on the basis of this additional assumption, it must be true that $x \in A \cap \overline{B}$, that is, $x \in A$ and $x \in \overline{B}$. We already know $x \in A$, by our initial assumption in the proof. As for $x \in \overline{B}$, that follows immediately from our additional assumption $x \notin B$.

If a desired conclusion has more than two alternatives, the strategy suggested by [6] is generalized as follows: Assume the negation of all but one of the alternative conclusions and, on that basis, try to prove that the remaining one must be true. We illustrate this in Example 16.

**0102153menu-proof-examplesExample 16** Prove that for all sets $A$ and $B$, if $A \times B = B \times A$, then either $A = \emptyset$ or $B = \emptyset$ or $A = B$.

*0102153Sketch of Proof:* [Note first that the notation $A \times B$ refers to the *cartesian product* of sets $A$ and $B$, defined as the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. This definition is the basis of the content of Chapter 6 of the text, on relations.] Let $A$ and $B$ be any sets such that $A \times B = B \times A$. Assume further that $A \neq \emptyset$ and $B \neq \emptyset$. With these two additional assumptions, our goal then becomes to prove that the third alternative $A = B$ must be true. The remainder of this proof is left as an exercise.

The following exercises provide the opportunity to use the strategy suggested by the equivalence [6].

- Complete the proof in Example 16.
- Prove that if $A$, $B$, and $C$ are any sets such that $A \times B = A \times C$, then either $A = \emptyset$ or $B = C$.
- Prove that if $A$ and $B$ are any sets such that $A \times B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.

- Prove that for all sets $X$, $Y$, and $Z$, $(X \cup Y) \cap Z \subseteq X \cup (Y \cap Z)$.

## 0102153menu-proofwriting255204512. Remarks on additional methods of proof

Not all propositions we may wish to prove have conclusions involving the form $\forall x[P(x) \rightarrow Q(x)]$. Nonetheless, beginning students who are able to write correctly the proofs called for in the exercises in Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."Section 1 of this Guide are well prepared to deal with the new issues that arise in writing other types of proofs. One reason for this is that many of the tactics (e.g., division into cases, specialization) and strategies (e.g., the choose method, indirect proof), highlighted in Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."Section 1, have application beyond proving propositions whose conclusion is of the form $\forall x[P(x) \rightarrow Q(x)]$. In this section, we discuss briefly two additional types of propositions.

### *0102153***2.1.** *0102153menu-proofwritingDeducing conclusions having the form "For every $x$, there exists $y$ such that $P(x, y)$."*

Many defining properties in mathematics have one of the forms $\exists x\, P(x)$ or $\forall x\, \exists y\, P(x, y)$. Elementary definitions of these types include:

(i) Let $m$ and $n$ be integers. We say that $m$ *divides* $n$, denoted $m|n$, if and only if *there exists* an integer $p$ such that $n = mp$. (Cf. Definition 1 in Section 2.3 of the text.)

(ii) A function $f$ is *onto*: $f$ is onto if and only if, *for every* $y$ in the codomain of $f$, *there exists* $x$ in the domain of $f$ such that $f(x) = y$. (Cf. Definition 7 in Section 1.6 of the text.)

(iii) A real number $x$ is said to be *rational* if and only if *there exist* integers $p$ and $q$, with $q \neq 0$, such that $x = p/q$.

Many important mathematical propositions whose proofs should be within the capabilities of students working through this Guide have as their conclusion a statement involving one of the preceding definitions. Some examples are:

(a) Prove that if $m$, $n$, and $p$ are integers such that $m$ divides $n$ and $m$ divides $p$, then $m$ divides $n + p$.

(b) Prove that if functions $f$ and $g$, having the real numbers as their domain and codomain, are both onto, then their composition $f \circ g$ is also onto.

(c) Prove that if $x$ and $y$ are rational, then $xy$ and $x + y$ are rational.

The new issue involved in proving propositions like (a)–(c) is *existence*. At a key point of each of these proofs, we must "produce," or define, an appropriate object of the type whose existence is asserted in the desired conclusion. In doing this, it is important to realize that, for a conclusion of the form $\forall x\, \exists y\, P(x, y)$, the $y$ whose existence is to be proved usually *depends on the given* $x$; we should expect it to be defined in terms of $x$ or else in terms of some other object that is defined in terms of $x$. This principle is demonstrated in Examples 17 and 18, which follow.

**0102153menu-proof-examplesExample 17** Prove the first part of prop:cProposition (c): If $x$ and $y$ are rational, then $xy$ is rational.

*0102153Discussion.* Assume that real numbers $x$ and $y$ are rational. To prove that their product $xy$ is rational, we must show that $xy = p/q$, where $p$ and $q$ are integers with $q \neq 0$. Our job in this proof is to produce, literally to build, the integers $p$ and $q$ whose quotient $p/q$ equals $xy$. As in most proofs, once the argument is set up, we must next assess what we have available to work with. In the case of a proof of existence, this includes asking whether what we have to work with provides any "building blocks." We have at our disposal only the hypotheses that $x$ and $y$ are rational. This means we can state that there exist integers $p_1$ and $q_1$, with $q_1 \neq 0$, such that $x = p_1/q_1$; and there exist integers $p_2$ and $q_2$, with $q_2 \neq 0$, such that $y = p_2/q_2$. We note that, therefore, $xy = (p_1/q_1)(p_2/q_2)$, which, by rules of algebra, equals $(p_1p_2)/(q_1q_2)$. Noting that $p_1p_2$ and $q_1q_2$ are necessarily integers and that $q_1q_2 \neq 0$ (Why?), we declare that $p = p_1p_2$ and $q = q_1q_2$ are the required integers.

**0102153menu-proof-examplesExample 18** Prove prop:bProposition (b): If functions $f$ and $g$, having the real numbers as their domain and codomain, are both onto, then their composition $f \circ g$ is also onto.

*0102153Proof:* Assume that the functions $f$ and $g$ are onto. To prove that their composition $f \circ g$ is onto, let $z$ be an arbitrary real number. We must prove that there exists $x \in \mathbf{R}$ such that $(f \circ g)(x) = z$. Now since $f$ is onto, we know that, corresponding to the given $z$, there exists a real number $y$ such that $z = f(y)$. Next, since $g$ is onto, then corresponding to this $y$, there must exist a real number $w$ such that $y = g(w)$. Note therefore that $z = f(y) = f(g(w)) = (f \circ g)(w)$. Hence our choice of the desired real number $x$ becomes evident, namely choose $x = w$.

Theorem 1 in Section 2.3 of the text contains several propositions related to defiDefinition (i), including a proof of prop:aProposition (a). You should study that proof, noting its similarities to the proofs in Examples 17 and 18, and then attempt the following exercises.

- Prove parts 2 and 3 of Theorem 1 in Section 2.3 of the text.
- Prove that for every integer $n$, there exists an integer $m$ such that $m|n$.
- Prove that for every positive integer $m$, there exists a positive integer $n$ such that $m|n$.
- Prove the second part of prop:cProposition (c): If $x$ and $y$ are rational, then $x + y$ is rational.
- Prove that if $f$ and $g$ are functions having the real numbers as their domain and codomain, and if $f \circ g$ is onto, then $f$ is onto.

### *0102153***2.2.** *0102153menu-proofwritingProof by mathematical induction*

We use mathematical induction to prove a proposition whose conclusion has the form $\forall n\, P(n)$, where *$n$ is a positive integer* (or, sometimes, a nonnegative integer). Thus proof by induction is an appropriate approach when the universe of discourse for a predicate quantified by "for every" is the set of all positive integers. If you review earlier sections of this Guide, you will note that this has not usually been the case in most of the examples and exercises covered, so induction would not have been an appropriate approach at those earlier stages.

Note that the *inductive step* in every proof by mathematical induction involves a proposition of the form $\forall n[P(n) \rightarrow Q(n)]$, where $Q(n)$ is $P(n+1)$. Thus the basic approach to be taken in the second part of a proof by induction is the same approach that was emphasized throughout Deducing conclusions having the form "For every $x$, if $P(x)$, then $Q(x)$."Section 1 of this Guide, namely the choose method. We start by letting $n$ be an arbitrary positive integer for which it is assumed that $P(n)$ is true. We must prove, on the basis of that assumption and whatever else is available (e.g., hypotheses), that $P(n + 1)$ is also true.

For more on mathematical induction, see Section 3.2 of the text.

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