Comments, hints, and some solutions for exam dat401 DM 020320

This is not a model solution!

1a Solution: 3 (not 5, which is the most popular wrong answer)

1b Note that the graph must be connected (less than one third got this one right).

1d The event '$\mathrm{diam}(G(n,p)) = 1$' is the same as the event '$G(n,p) = K_n$.'

1e Solution: $1 - p$. Many students use a very long argument which involves counting the number of graphs with $d(v_1, v_2) > 1$. This is impressive but unnecessary: the distance is 1 if and only if the edge $\{v_1, v_2\}$ is present, which happens (by definition) with probability p.

1f Solution: $(1 - p^2)^{n-3}$.

1g Hint: $p\big(\mathrm{diam}(G(n,p)) = 2\big) = 1 - p\big(\mathrm{diam}(G(n,p)) = 1\big) - p\big(\mathrm{diam}(G(n,p)) > 2\big)$. For the last expression use 1f and $P(A \cup B) \leq P(A) + P(B)$.

2a There are 15. (Few students found them all).

2b Solution: Yes.

2c Solution: Yes.

2d The answer is yes. A 'formal proof' is an opportunity to show your familiarity with the *language of maths* — so please don't write nonsense like '$a \wedge b \wedge c \in T_i$' when you mean '$\{a, b, c\} \subseteq T_i$'. Note also that you proof needs to use the disjointness of the T_i somewhere, otherwise it's incomplete.

2e Many students' arguments would benefit from the phrase 'by the Sum Rule'. Here's the most popular argument (which is sufficient): "A partition of S places the elements into a number of subsets $T_1, \dots, T_k$; at least $k \geq 1$ (in which case $T_1 = S$) and at most $k \leq n$ (in which case $T_i = \{i\}$). By the Sum Rule we can count the total number of partitions of S (of any size) by for each k ($1 \leq k \leq n$) counting the number of ways to partition S into exactly k subsets, which is given by $\left\{ {n \atop k} \right\}$." Note that this answer is quite precise in explaining the bounds on the sum (k from 1 to n).

2f *Hint*: Consider the set into which '$n + 1$' has been placed.

3a Should take 26 moves.

3b Solution: $T_n = 3T_{n-1} + 2$, $T_1 = 2$. *Don't* forget the T_1 case!

3d Many students solve this by induction, which is really cute! The solution I had in mind simply counts the number of different legal arrangements (*hint*: place all discs on the pegs, starting with the largest) and notes that this number is $1 + T_n$.