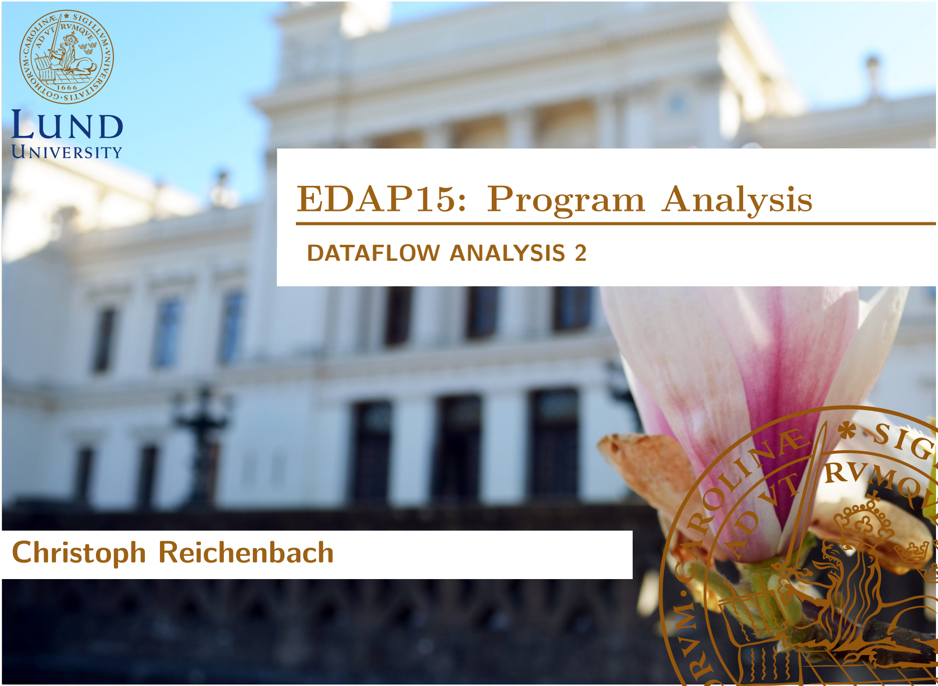


EDAP15: Program Analysis

DATAFLOW ANALYSIS 2

Christoph Reichenbach

Welcome back!

- No new homework this week
- Quiz deadline clarification by Friday
- Questions?

Data Flow Analysis on CFGs

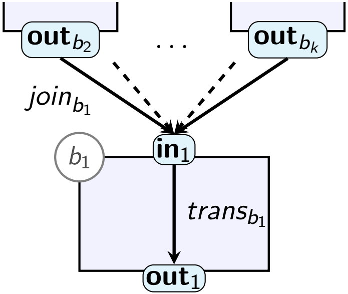
- $join_b$: *Join Function*
- $trans_b$: *Transfer Function*
- $\mathbf{in}_b$: knowledge at entrance of b

$$\mathbf{in}_{b_1} = join_{b_1}(\mathbf{out}_{b_2}, \ldots, \mathbf{out}_{b_k})$$

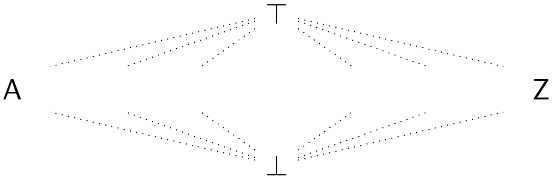
- $\mathbf{out}_b$: knowledge at exit of b

$$\mathbf{out}_{b_1} = trans_{b_1}(\mathbf{in}_{b_1})$$

- *Forward Analysis* shown here
- *Backward Analysis*: flip edge direction



Join and Transfer Functions



- L: Abstract Domain
  - Ordered by $(\sqsubseteq) \subseteq L \times L$

    $\top \in L$ for all x: $x \sqsubseteq \top$ Top element

    $\bot \in L$ for all x: $\bot \sqsubseteq x$ Bottom element (optional)
- $trans_b : L \to L$
  - monotonic
- $join_b : L \times \ldots \times L \to L$
  - pointwise monotonic

$$\begin{array}{ccc} x & \sqsubseteq & y \\ & \Downarrow & \\ trans_b(x) & \sqsubseteq & trans_b(y) \end{array}$$

$$\begin{array}{ccc} x & \sqsubseteq & y \\ & \Downarrow & \\ join_b(z_1, \ldots, z_k, x, \ldots, z_n) & \sqsubseteq & join_b(z_1, \ldots, z_k, y, \ldots, z_n) \end{array}$$

Monotone Frameworks

Monotone Framework	Lattice
Abstract Domain	$L = \langle \mathcal{L}, \sqsubseteq, \sqcap, \sqcup \rangle$
$join_b(x_1, \ldots, x_n)$	$x_1 \sqcup \ldots \sqcup x_n$
	$x \sqcap y$ (Used next week)
'Unknown' start value	$\bot$
'Could be anything' end value	$\top$

- *Monotone Frameworks* (Killdall '77):
  - Lattice L of *finite height* (= satisfies Ascending Chain Condition)
  - Monotone $trans_b$
    - 'compatible' with semantics

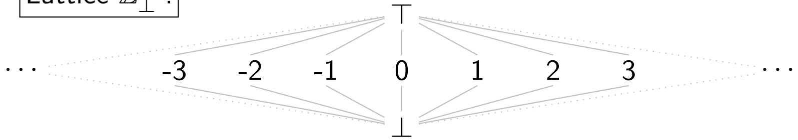
$\Longrightarrow$ Data flow analysis with Soundness and Termination guarantee

- Don't need $\sqcap$ *yet*, so technically we can get by with a *Semilattice*.

Product Lattices and Values

- Consider **Constant Propagation + Folding** on lattice $\mathbb{Z}_\perp^\top$

Lattice $\mathbb{Z}_\perp^\top$:



- Program with three variables: `x`, `y`, `z`
- Lattice value that represents the outcome of this code:

b_0

```
var x := 0
var y := 1
var z := 2
```

$$\langle \overset{\texttt{x}}{\mathbf{0}}, \overset{\texttt{y}}{\mathbf{1}}, \overset{\texttt{z}}{\mathbf{2}} \rangle$$

- Value in $\mathbb{Z}_\perp^\top \times \mathbb{Z}_\perp^\top \times \mathbb{Z}_\perp^\top$

Transfer Functions and Updates

- With n program variables, abstract domain is $(\mathbb{Z}_\bot^\top)^n$
- For each CFG node b_i:
  - Transfer functions $trans_i = [\![b_i]\!]$
  - $[\![b_i]\!]$ update lattice elements (monotonically):

$$[\![b_i]\!] : (\mathbb{Z}_\bot^\top)^n \to (\mathbb{Z}_\bot^\top)^n$$

- For readability: denote $\sigma \in (\mathbb{Z}_\bot^\top)^n$ as *finite maps* (i.e., write $[\texttt{varname} \mapsto \textit{abstract value}]$):

b_0

```
var x := 0
var y := 0
var z := 0
```

$$trans_0(\sigma) = [\![b_0]\!](\sigma) = [\begin{array}{lll} x & \mapsto & 0; \\ y & \mapsto & 0; \\ z & \mapsto & 0 \,] \end{array}$$

Simplified notation

b_1

```
y := 7
z := z + 1
```

$$trans_1(\sigma) = [\![b_1]\!](\sigma) = [\ x \mapsto \sigma(x);\ y \mapsto 7;\ z \mapsto \sigma(z) + 1\ ]$$

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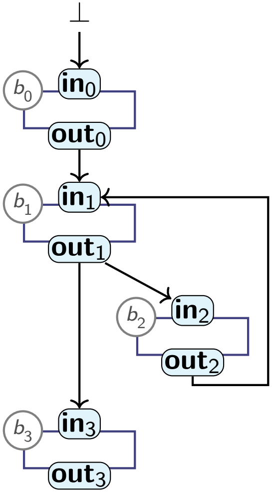
$$trans_1 = [\![b_1]\!] = [\ y \mapsto 7;\ z \mapsto z + 1\ ]$$

Formalising a Naïve Algorithm

$$
\begin{aligned}
\mathbf{out}_0 &= \mathit{trans}_0(\mathbf{in}_0) &&= \mathit{trans}_0(\bot) \\
\mathbf{out}_1 &= \mathit{trans}_1(\mathbf{in}_1) &&= \mathit{trans}_1(\mathbf{out}_0 \sqcup \mathbf{out}_2) \\
\mathbf{out}_2 &= \mathit{trans}_1(\mathbf{in}_2) &&= \mathit{trans}_2(\mathbf{out}_1) \\
\mathbf{out}_3 &= \mathit{trans}_1(\mathbf{in}_3) &&= \mathit{trans}_3(\mathbf{out}_1)
\end{aligned}
$$



- Lattices $\mathbf{out}_0 : L_0, \ldots, \mathbf{out}_3 : L_3$
- Can build lattice for entire program:
  - $L_{0\ldots3} = L_0 \times L_1 \times L_2 \times L_3$
  - $\bot_{0\ldots3} = \langle \bot_0, \bot_1, \bot_2, \bot_3 \rangle$
  - Monotone transfer function:

$$
\mathit{trans}_{0\ldots3}(\langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle) = \left\langle \begin{array}{l} \mathit{trans}_0(\sigma_0), \\ \mathit{trans}_1(\sigma_0 \sqcup \sigma_2), \\ \mathit{trans}_2(\sigma_1), \\ \mathit{trans}_3(\sigma_1) \end{array} \right\rangle
$$

Reaching a Solution

- Abstract approach:
  - Program P:
    - "Program Lattice" $L_P = L_0 \times \cdots \times L_n$
    - $\bot_P = \langle \bot_0, \ldots, \bot_n \rangle$: initial analysis state
    - $trans_P$: Compute one step of naïve analysis
  - Repeat $trans_P$ until solution $fp_\bot$:

$$fp_\bot = trans_P^n(\bot_P)$$

  - n is the minimum number of steps until result does not change any more (= we have a solution)
- $fp_\bot$ is *Fixpoint* of $trans_P$:

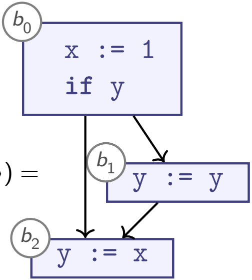
$$fp_\bot = trans_P^+(fp_\bot)$$

- Fixpoint exists in L_P **iff** $trans_P^k$ satisfies Ascending Chain Condition

Cousot & Cousot (1979), based on Kleene (1952), based on Knaster & Tarski (1933)

Naïve Iteration

Analysis on
$\mathbb{Z}_\perp^\top \times \mathbb{Z}_\perp^\top$



$$trans_0([x \mapsto x, y \mapsto y]) = [x \mapsto \mathbf{1}, y \mapsto y]$$

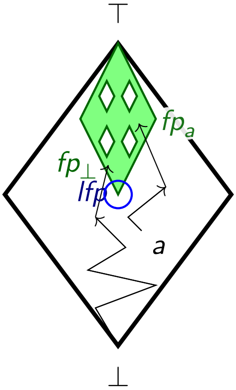
$$trans_P(\langle \mathbf{in}_0, \mathbf{out}_0, \mathbf{out}_1, \mathbf{out}_2 \rangle) = \left\langle \begin{array}{l} \mathbf{in}_0, \\ trans_0(\mathbf{in}_0), \\ trans_1(\mathbf{out}_0), \\ trans_2(\mathbf{out}_0 \sqcup \mathbf{out}_1) \end{array} \right\rangle$$

$$trans_1(\sigma) = \sigma$$

$$trans_2([x \mapsto \mathbf{x}, y \mapsto y]) = [x \mapsto x, y \mapsto \mathbf{x}]$$

	$\perp_P$	$trans_P^1(\perp_P)$	$trans_P^2(\perp_P)$	$trans_P^3(\perp_P)$
$\mathbf{in}_0$	$\perp$	$\perp$	$\perp$	$\perp$
$\mathbf{out}_0$	$\perp$	$x \mapsto 1$	$x \mapsto 1$	$x \mapsto 1$
$\mathbf{out}_1$	$\perp$	$\perp$	$x \mapsto 1$	$x \mapsto 1$
$\mathbf{out}_2$	$\perp$	$\perp$	$x \mapsto 1, y \mapsto 1$	$x \mapsto 1, y \mapsto 1$

Fixpoints



- Repeat $trans_P$ until we reach a fixpoint
- Can start from *any* point a
- Multiple fixpoints possible
  - Each is a *sound* solution (for *compatible* transfer functions)
  - Form a lattice (Knaster-Tarski, 1933)
- Least Fixpoint: Highest Precision

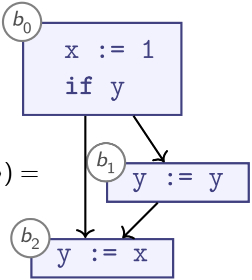
Starting from *any* point? Even ⊤?

Naïve Iteration

Analysis on
$\mathbb{Z}_\bot^\top \times \mathbb{Z}_\bot^\top$

$trans_P(\langle \mathbf{in}_0, \mathbf{out}_0, \mathbf{out}_1, \mathbf{out}_2 \rangle) =$

$$\left\langle \begin{array}{l} \mathbf{in}_0, \\ trans_0(\mathbf{in}_0), \\ trans_1(\mathbf{out}_0), \\ trans_2(\mathbf{out}_0 \sqcup \mathbf{out}_1) \end{array} \right\rangle$$



$$trans_0([x \mapsto x,\ y \mapsto y]) = [x \mapsto \mathbf{1},\ y \mapsto y]$$

$$trans_1(\sigma) = \sigma$$

$$trans_2([x \mapsto \mathbf{x},\ y \mapsto y]) = [x \mapsto x,\ y \mapsto \mathbf{x}]$$

	$\top_P$	$trans_P^1(\top_P)$	$trans_P^2(\top_P)$	$trans_P^3(\top_P)$
$\mathbf{in}_0$	$\top$	$\top$	$\top$	$\top$
$\mathbf{out}_0$	$\top$	$x \mapsto 1, y \mapsto \top$	$x \mapsto 1, y \mapsto \top$	$x \mapsto 1, y \mapsto \top$
$\mathbf{out}_1$	$\top$	$\top$	$x \mapsto 1, y \mapsto \top$	$x \mapsto 1, y \mapsto \top$
$\mathbf{out}_2$	$\top$	$\top$	$\top$	$x \mapsto 1, y \mapsto 1$

Starting from $\bot$ vs $\top$

- Starting from $\top$ works fine
- *Naïve iteration* can increase precision of imprecise starting assumptions

Summary

- **Monotone Frameworks**:
  - Combine:
    - Monotone transfer functions $trans_b$
    - Finite-Height Lattices

$$join_b(\sigma_1, \ldots, \sigma_k) = \sigma_1 \sqcup \ldots \sqcup \sigma_k$$

  - Guarantee:
    - Termination
    - Soundness
- With Monotone Frameworks, iterating $trans_b$ and $join_b$ produces **Fixpoint** (or *Fixed Point*)
  - Works from *any* starting point, possibly different fixpoint
    - Fixpoints form **Fixpoint Lattice**
    - **Least Fixpoint** (Bottom element) is *most precise solution*
- (Soundness only if $trans_b$ are *compatible*)

An Algorithm for Fixpoints

- So far: naïve algorithm for computing fixpoint
  - Produces a fixpoint
  - Keeps iterating *all* $trans_b$ / $join_b$ functions, even if nothing changed
- Optimise processing with *worklist*
  - Set-like datastructure:
    - **add** element (if not already present)
    - **contains** test: is element present?
    - **pop** element: remove and return one element
  - Tracks *what's left to be done*

$\Rightarrow$ "MFP" (Minimal Fixed Point) Algorithm
(*Does not always produce best result* $\rightarrow$ *will see later today*)

Example: Constant Propagation + Folding with Size Limit

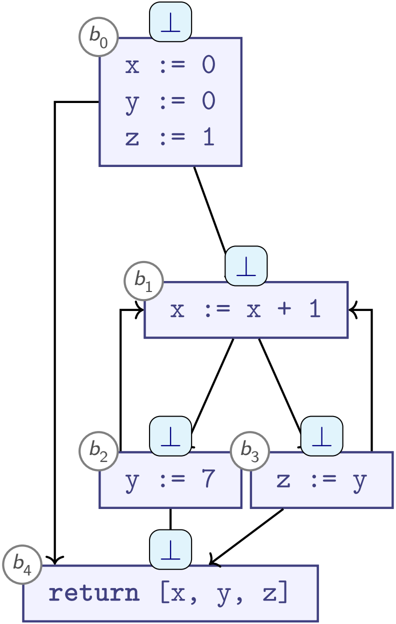
- For next example, we use the following lattice:
- Elements:

$$\mathbb{Z} \cup \{S \mid S \subseteq \mathbb{Z} \text{ and } \#S \leq 3\}$$

- Relations and operations:
  - $a \sqsubseteq b \iff a \subseteq b$
  - $\sqcup = \cup$
  - $\sqcap = \cap$
  - $\top = \mathbb{Z}$
  - $\bot = \emptyset$
- Lattice has finite height
  Longest chains have five elements:

$$\emptyset \sqsubseteq \{x\} \sqsubseteq \{x, y\} \sqsubseteq \{x, y, z\} \sqsubseteq \mathbb{Z}$$

MFP Example:



		$trans_b$		
b	inputs	x	y	z
b_0	$\emptyset$	0	0	1
b_1	$\{b_0, b_2, b_3\}$	$x + 1$	y	z
b_2	$\{b_1\}$	x	7	z
b_3	$\{b_1\}$	x	y	y
b_4	$\{b_0, b_2, b_3\}$	x	y	z

$$join_{b_i}(\sigma_1, \sigma_2) = [\ x \mapsto \sigma_1(x) \cup \sigma_2(x),\ y \mapsto \sigma_1(y) \cup \sigma_2(y),\ z \mapsto \sigma_1(z) \cup \sigma_2(z)\ ]$$

Worklist

$b_0 \to b_1$
$b_0 \to b_4$
$b_1 \to b_2$
$b_1 \to b_3$
$b_2 \to b_4$
$b_2 \to b_1$
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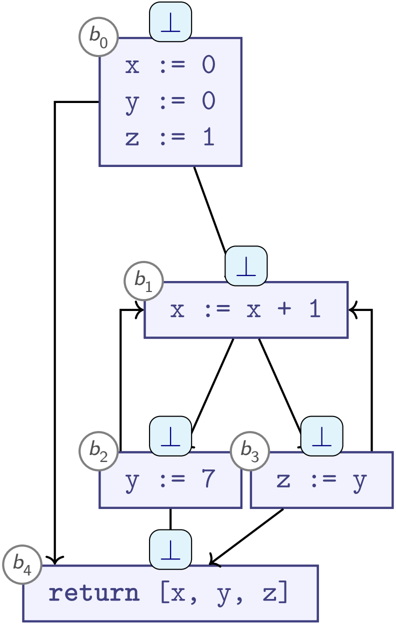
MFP Example:



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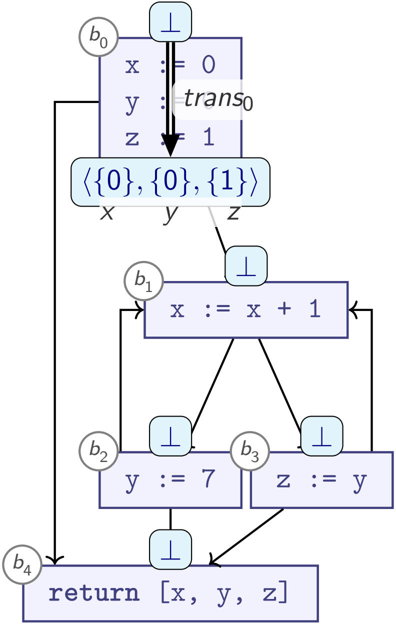

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For edge $b_o \to b_i$:

- $\mathbf{out}_o := trans_o(\mathbf{in}_o)$

Worklist

- $b_0 \to b_1$
- $b_0 \to b_4$
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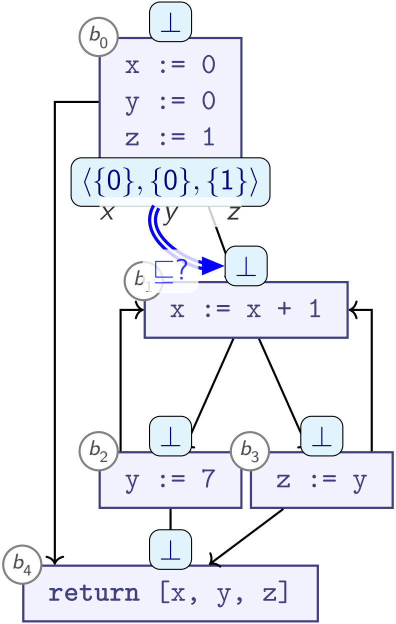

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For edge $b_o \to b_i$:

- $\mathbf{out}_o := trans_o(\mathbf{in}_o)$
- Is $\mathbf{out}_o \not\sqsubseteq \mathbf{in}_i$?

Worklist

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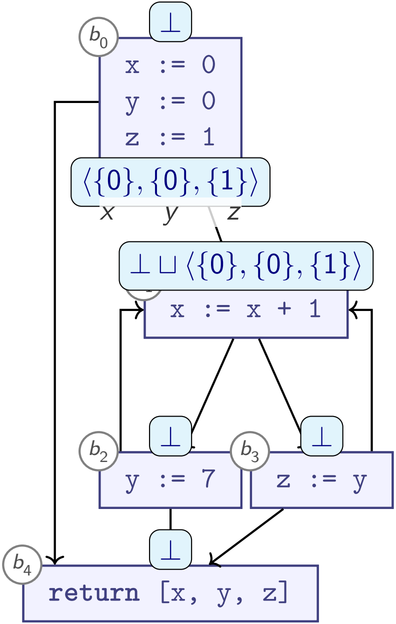

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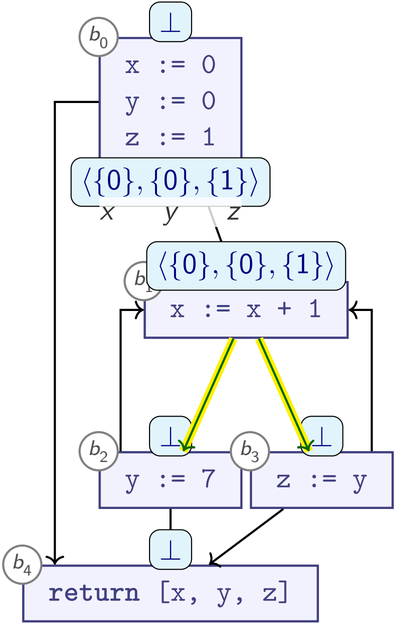

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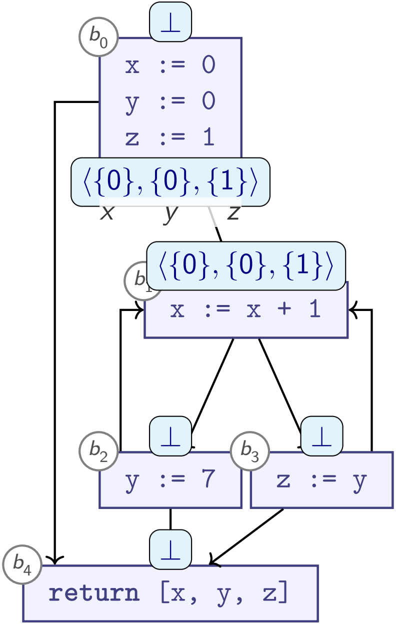

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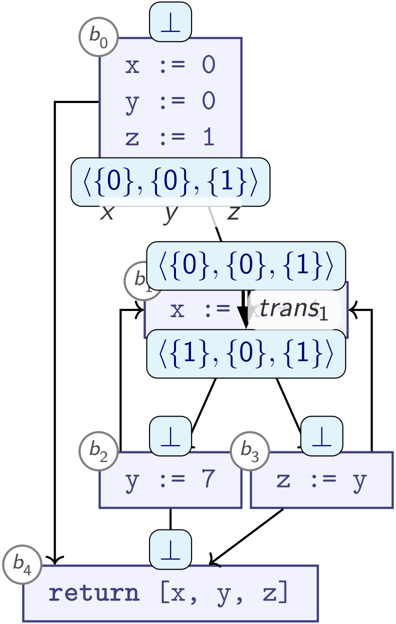
MFP Example:



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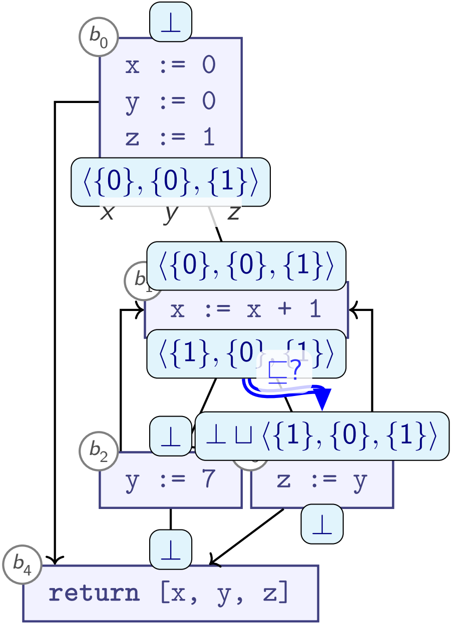
$b_3 \to b_1$

MFP Example:



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b_0	$\emptyset$	0	0	1
b_1	$\{b_0, b_2, b_3\}$	$x + 1$	y	z
b_2	$\{b_1\}$	x	7	z
b_3	$\{b_1\}$	x	y	y
b_4	$\{b_0, b_2, b_3\}$	x	y	z

$$join_{b_i}(\sigma_1, \sigma_2) = [\; x \mapsto \sigma_1(x) \cup \sigma_2(x),\; y \mapsto \sigma_1(y) \cup \sigma_2(y),\; z \mapsto \sigma_1(z) \cup \sigma_2(z) \;]$$

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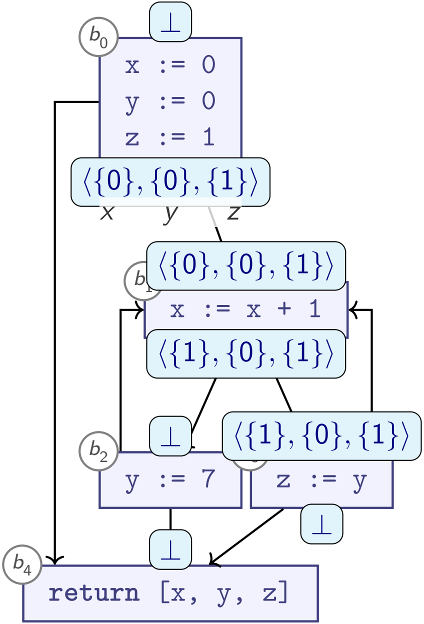
Worklist

$b_0 \rightarrow b_4$
$b_1 \rightarrow b_2$
$b_1 \rightarrow b_3$
$b_2 \rightarrow b_4$
$b_2 \rightarrow b_1$
$b_3 \rightarrow b_4$
$b_3 \rightarrow b_1$

MFP Example:



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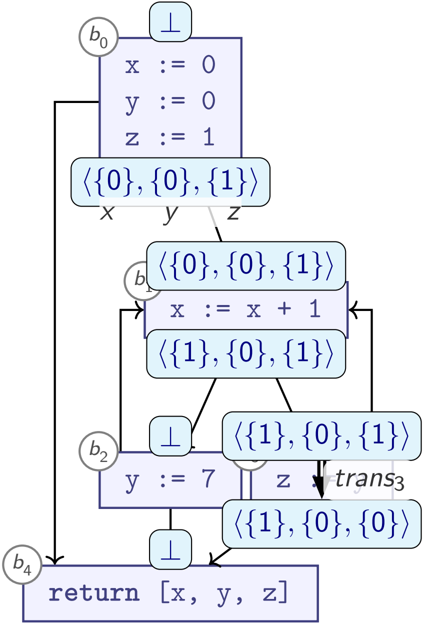

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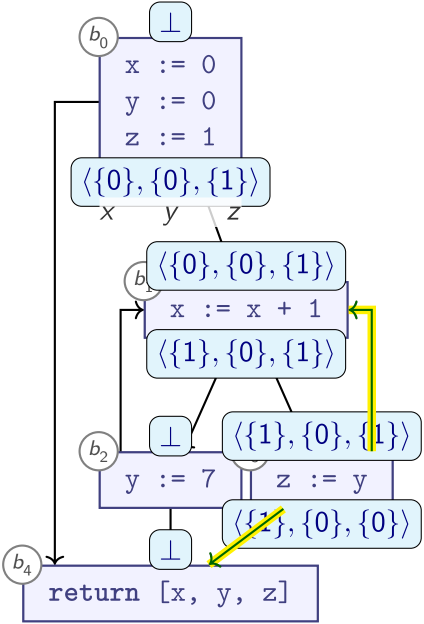

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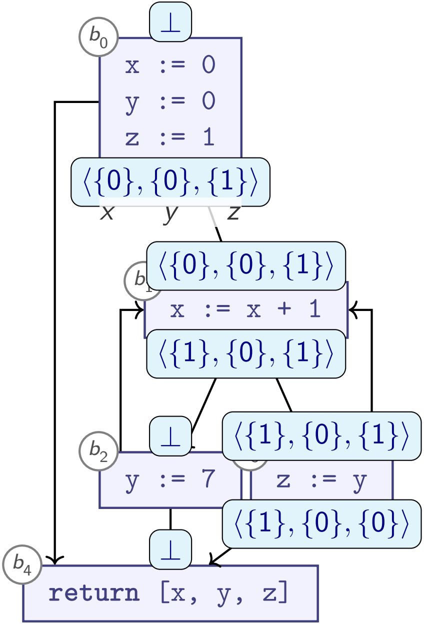

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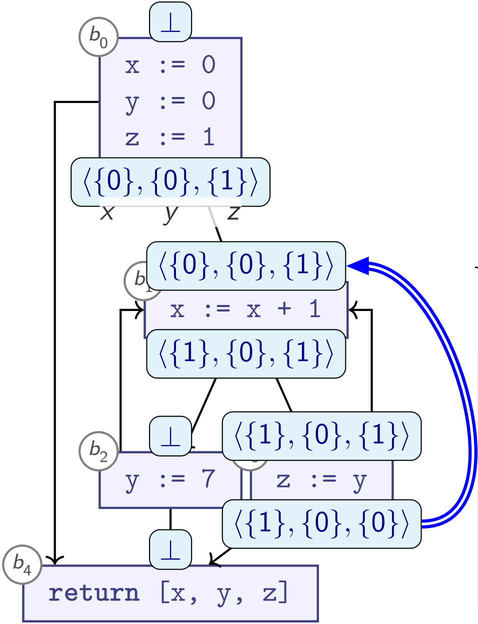

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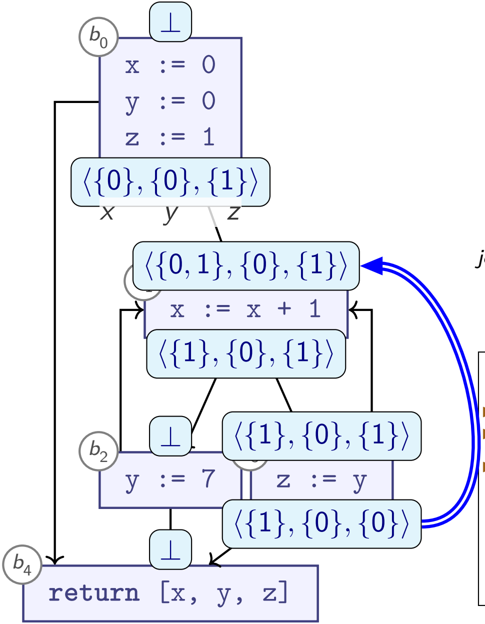

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Worklist

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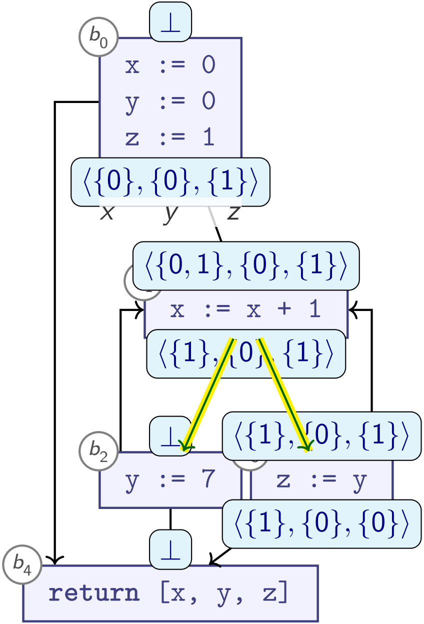

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- Is **out**$_o \not\sqsubseteq$ **in**$_i$?
- **Yes**:
  - **in**$_i$:= **in**$_i \sqcup$ **out**$_o$
  - Add all outgoing edges from (if no

Re-add previously removed edge

Worklist

$b_0 \to b_4$
$b_1 \to b_2$

$b_2 \to b_4$
$b_2 \to b_1$
$b_3 \to b_4$

$b_1 \to b_3$

The MFP Algorithm

```
Procedure MFP(⊥, ⊔, ⊑, CFG, trans_, is-backward):
begin
  if is-backward then reverse edges(CFG);
  worklist := edges(CFG); -- edges that we need to look at
  foreach n ∈ nodes(CFG) do
    in[n] := ⊥;           -- state of the analysis
  done
  while not empty(worklist) do
    ⟨n, n′⟩ := pop(worklist);     -- Edge n → n′
    out_n := trans_n(in[n]);     -- Consider caching out_n
    if out_n ⋢ in[n′] then begin
      in[n′] := in[n′] ⊔ out_n;
      foreach n′′ ∈ successor-nodes(CFG, n′) do
        push(worklist, ⟨n′, n′′⟩);
      done
    end
  done
  return in;
end
```

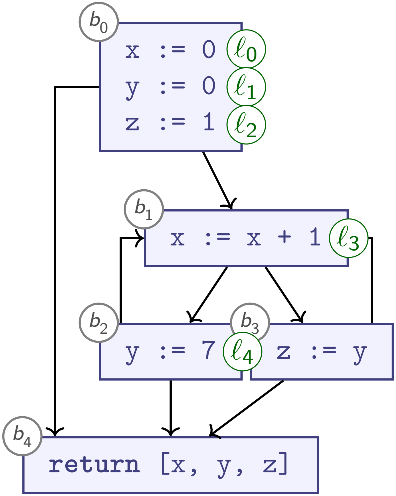
Worklist allows focussing effort!

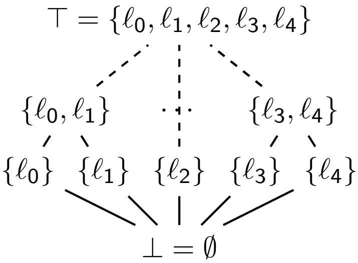
Summary: MFP Algorithm

- **Product Lattice** allows analysing multiple variables at once
- Compute data flow analysis:
  - Initialise all nodes with $\bot$
  - Repeat until nothing changes any more:
    - Merge updates monotonically via $\sqcup$
    - Apply transfer function
    - Propagate changes along control flow graph
- Compute **fixpoint**
- Use **worklist** to increase efficiency
- Distinction: Forward/Backward analyses

MFP revisited

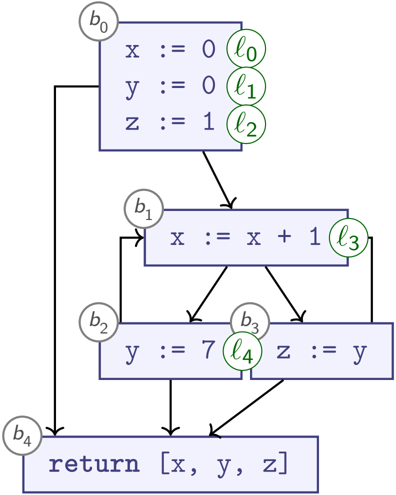
Reaching Definitions analysis: which *program location* might a given value be coming from?





- All subsets of $\{\ell_0, \dots, \ell_4\}$
- Finite height
- $\sqcup = \cup$

MFP revisited: Transfer Functions



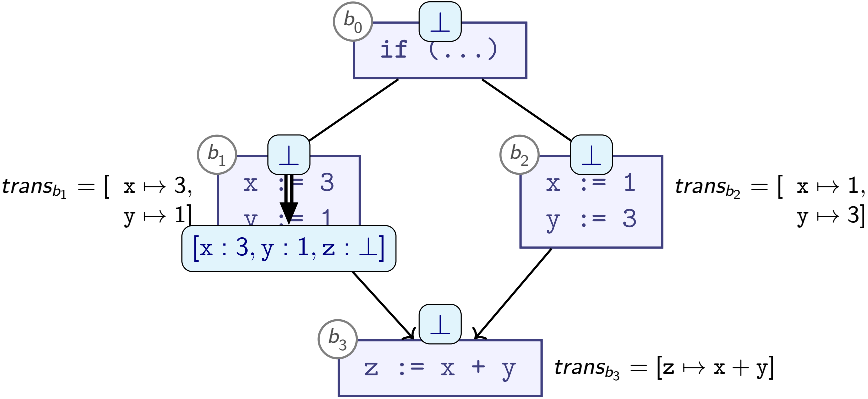
$$
\begin{aligned}
trans_{b_0} &= [\; x \mapsto \{\ell_0\},\; y \mapsto \{\ell_1\},\; z \mapsto \{\ell_2\}] \\
trans_{b_1} &= [x \mapsto \{\ell_3\}] \\
trans_{b_2} &= [y \mapsto \{\ell_4\}] \\
trans_{b_3} &= [z \mapsto y]
\end{aligned}
$$

MFP solution at b_4

$$
\begin{aligned}
x &\mapsto \{\ell_0, \ell_3\} \\
y &\mapsto \{\ell_1, \ell_4\} \\
z &\mapsto \{\ell_1, \ell_2, \ell_4\}
\end{aligned}
$$

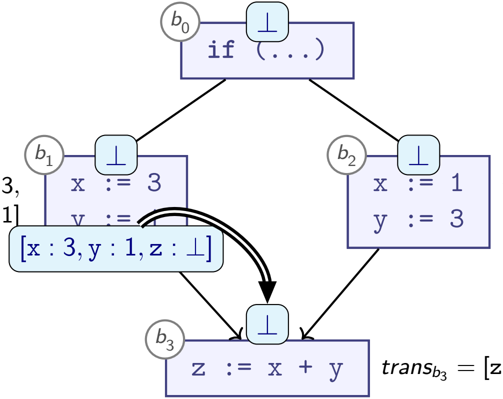
- Least Fixpoint!
- Do we always get LFP from MFP?

Another Example



- Lattice: $\mathbb{Z}_\bot^\top$

Another Example



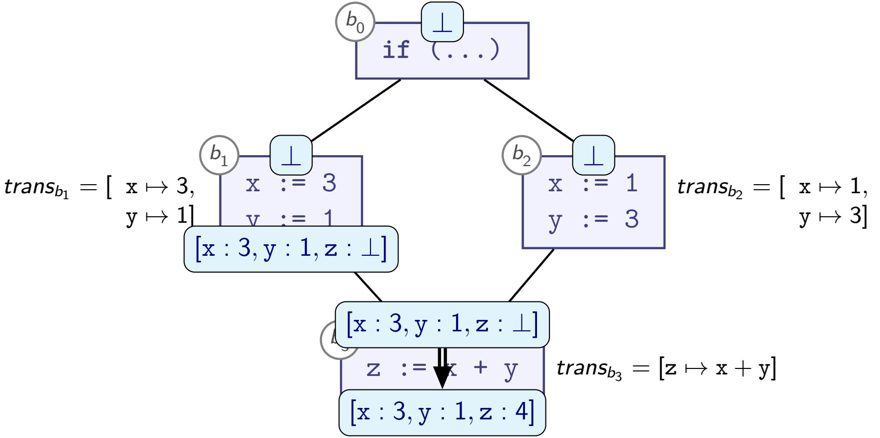
$trans_{b_1} = [\ \mathtt{x} \mapsto 3,\ \mathtt{y} \mapsto 1]$

$trans_{b_2} = [\ \mathtt{x} \mapsto 1,\ \mathtt{y} \mapsto 3]$

$trans_{b_3} = [\mathtt{z} \mapsto \mathtt{x} + \mathtt{y}]$

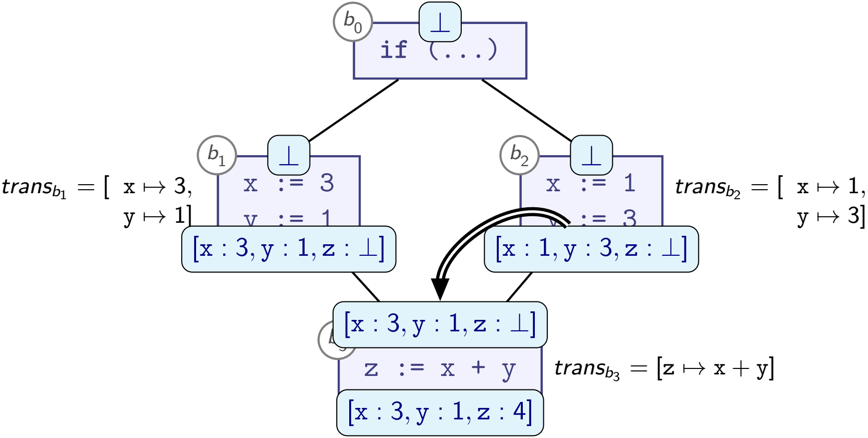
- Lattice: $\mathbb{Z}_\bot^\top$

Another Example



- Lattice: $\mathbb{Z}_{\perp}^{\top}$

Another Example



- Lattice: $\mathbb{Z}_\bot^\top$
  - $1 \sqcup 3 = \top = 3 \sqcup 1$

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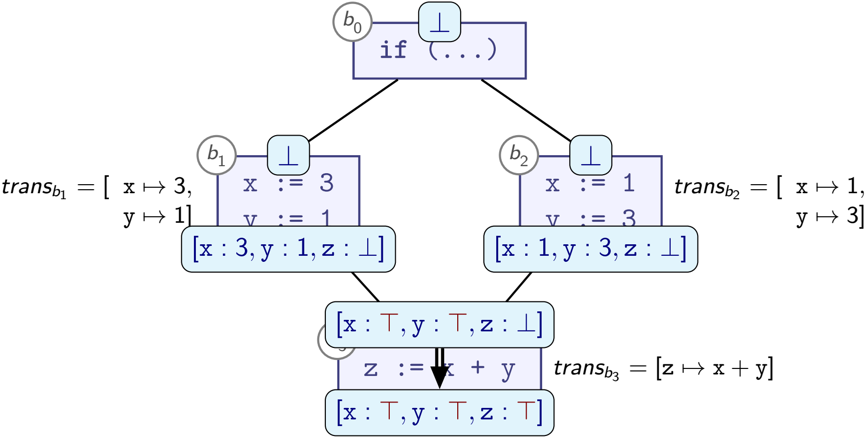
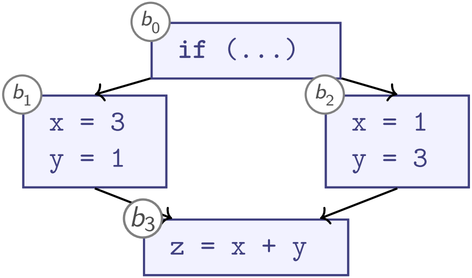


- Lattice: $\mathbb{Z}_\bot^\top$
  - $1 \sqcup 3 = \top = 3 \sqcup 1$
- MFP **does** compute the Least Fixpoint in our equations...
- ...but the fixpoint is worse than expected!

Execution paths



- Idea: Let's consider all *paths* through the program:

$$
\begin{aligned}
path_{b_0} &= \{()\} \\
path_{b_1} &= \{(b_0)\} \\
path_{b_2} &= \{(b_0)\} \\
path_{b_3} &= \{(b_0, b_1); (b_0, b_2)\}
\end{aligned}
$$

The MOP algorithm for Dataflow Analysis

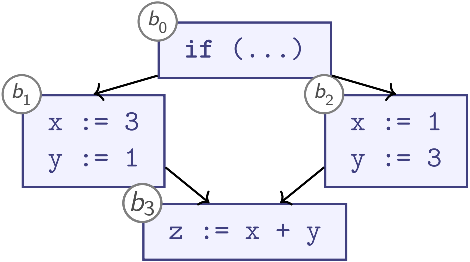
- Compute the MOP ('meet-over-all-paths') solution:
  - Iterate over all paths $(p_0, \ldots, p_k)$ in $path_{b_i}$
  - Compute *precise* result for that path
  - Merge (i.e., join, $\sqcup$) with all other precise results

$$\mathbf{out}_{b_i} = \bigsqcup_{(p_0,\ldots,p_k)\in path_{b_i}} trans_{b_i} \circ trans_{p_k} \circ \cdots \circ trans_{p_0}(\bot)$$

Notation: (*function composition*)

$$(f \circ g)(x) = f(g(x))$$

MOP vs MFP: Example



Transfer functions

$$\begin{aligned} trans_{b_0} &= id \\ trans_{b_1} &= [x \mapsto 3, y \mapsto 1] \\ trans_{b_2} &= [x \mapsto 1, y \mapsto 3] \\ trans_{b_3} &= [z \mapsto x + y] \end{aligned}$$

Paths

$$\begin{aligned} path_{b_0} &= \{()\} \\ path_{b_1} &= \{(b_0)\} \\ path_{b_2} &= \{(b_0)\} \\ path_{b_3} &= \{(b_0, b_1), (b_0, b_2)\} \end{aligned}$$

$$\begin{aligned} \mathbf{out}_{b_3} &= ([z \mapsto x + y][x \mapsto 3, y \mapsto 1](\bot)) \sqcup ([z \mapsto x + y][x \mapsto 1, y \mapsto 3](\bot)) \\ &= [z \mapsto 3 + 1, x \mapsto 3, y \mapsto 1] \sqcup [z \mapsto 1 + 3, x \mapsto 1, y \mapsto 3] \\ &= [z \mapsto 4, x \mapsto \top, y \mapsto \top] \end{aligned}$$

MOP vs MFP (1/2)

- In our example:

$$\begin{array}{ll} \textbf{MFP:} & [x \mapsto \top, \quad y \mapsto \top, \quad z \mapsto \top] \\ \textbf{MOP:} & [x \mapsto 4, \quad y \mapsto \top, \quad z \mapsto \top] \end{array}$$

- *Both are least fixed points*
- MOP and MFP use same transfer functions, same lattice
- However, MOP and MFP set up different equations

MOP vs MFP (2/2)

	MOP	MFP
Soundness	sound	sound
Precision	*maximal*	*sometimes lower*
Decidability	*undecidable*	*decidable*

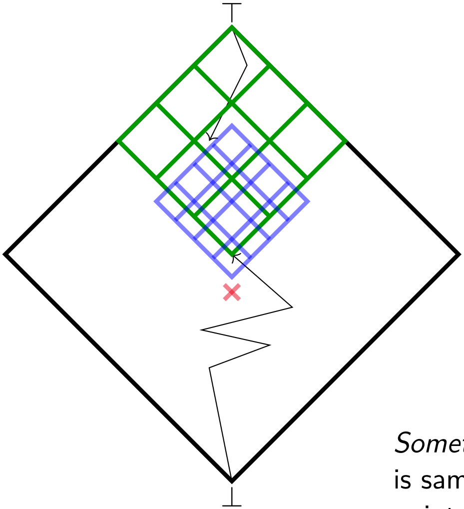
- MOP: Merge Over all Paths
  (Originally: "Meet Over all Paths", but we use the Join operator)
- MFP: Minimal Fixed Point

Summary

- $path_b$: Set of all paths from program start to b
- MOP: alternative to MFP (theoretically)
  - Termination not guaranteed
  - May be more precise
  - Idea:
    - Enumerate all paths to basic block
    - Compute transfer functions over paths individually
    - Join

Why is MFP *sometimes* as good as MOP?

MOP vs MFP Fixpoints



MFP

Naïve Iteration

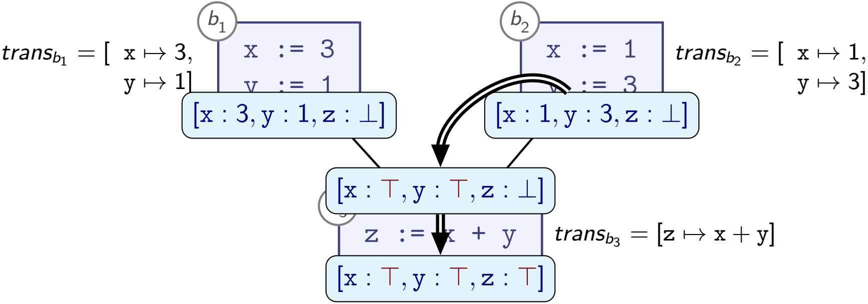
MOP

Sometimes MOP result is same as least fixed point for MFP...

Summary

- **MFP**
  - Avoids redundant computations
  - Fixpoint $\sqsupseteq$ starting point
- **Naïve fixpoint iteration**
  - Fixpoint may be *above* or *below* starting point
  - Can start with $\top$: always sound
    - Can control analysis cost with time budget
    - May lose precision with loops
- **MOP**
  - One fixpoint, no "starting point"
  - Maximal Precision
  - Undecidable in general
    - Used in *Model Checking*
- This list is not exhaustive
- All fixpoints are sound overapproximations

MFP vs the Least Fixpoint



- MFP is *sometimes* equal to MOP
- Challenge:

$$trans_b(x \sqcup y) \sqsupset trans_b(x) \sqcup trans_b(y)$$

- join-before-transfer: overapproximate before we can reconcile!

Distributive Frameworks

A *Monotone Framework* is:

- Lattice $L = \langle \mathcal{L}, \sqsubseteq, \sqcap, \sqcup \rangle$
- L has finite height (Ascending Chain Condition)
- All $trans_b$ are monotonic
- Guarantees a Fixpoint

A *Distributive Framework* is:

- A Monotone Framework, where additionally:
- $trans_b$ distributes over $\sqcup$:

$$trans_b(x \sqcup y) = trans_b(x) \sqcup trans_b(y)$$

  for all programs and all x, y, b
- Guarantees that MFP gives same Fixpoint as MOP

Distributive Problems

- Monotonic:

$$trans_b(x \sqcup y) \sqsupseteq trans_b(x) \sqcup trans_b(y)$$

- Distributive:

$$trans_b(x \sqcup y) = trans_b(x) \sqcup trans_b(y)$$

- Many analyses fit distributive framework
- Known *counter-example*: transfer functions on $\mathbb{Z}_\perp^\top$:
  - $[z \mapsto x + y]$
  - Generally:
    - depends on ≥ 2 independent inputs
    - can produce same output for different inputs

Summary

- **Distributive Frameworks** are *Monotone Frameworks* with additional property:

$$trans_b(x \sqcup y) = trans_b(x) \sqcup trans_b(y)$$

  for all programs and all x, y, b

- In Distributive Frameworks, MFP produces same least Fixpoint as for MOP
- Some analyses (Gen/Kill analyses, discussed later) are always distributive

Outlook

- Quiz deadline clarification by Friday
- Lab priority: Lab 1a for Friday

http://cs.lth.se/EDAP15