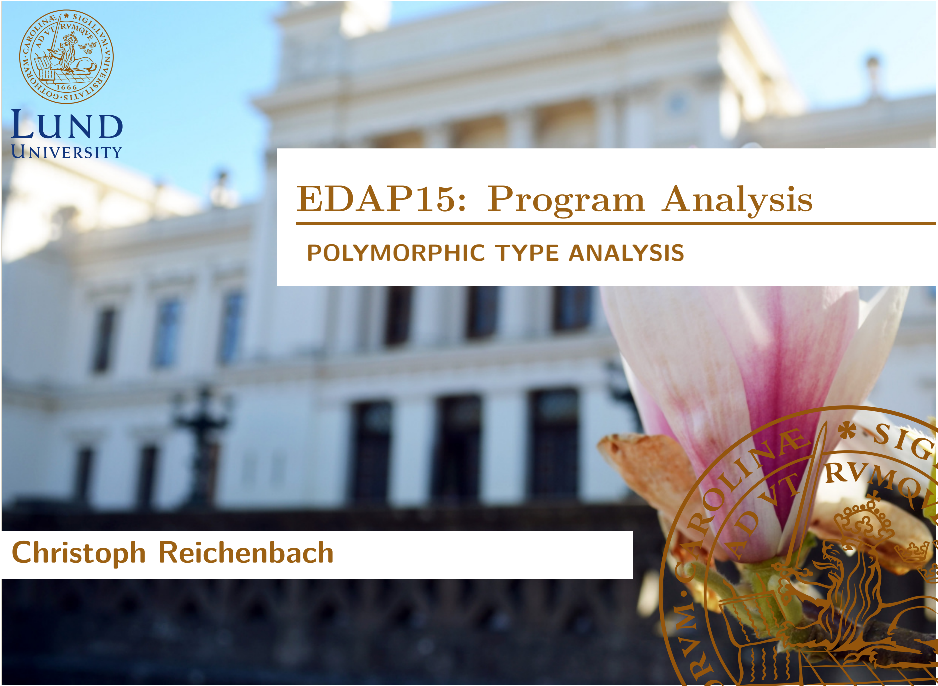

EDAP15: Program Analysis

POLYMORPHIC TYPE ANALYSIS

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Lecture Overview

Foundations

- 01 Foundations
- 02 Constructing Program Analyses in JastAdd

Static Analysis

Properties

- 03, 04 **Types**
- 05, 06, 07 Data Flow
- 08, 09 Memory

Control Flow

- 05 Intraprocedural
- 10 Interprocedural
- 11 Indirect

Dynamic Analysis

- 12 Instrumentation
- 13 Analysis

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Parametric Types and Principal Types

$$\begin{array}{rcl} ty & ::= & \textsc{Int} \\ & \mid & \textsc{Bool} \\ & \mid & \textsc{List}\ [\langle ty \rangle] \end{array}$$

- Challenge:
  - p : τ allows infinitely many types
- To make practical: find **Principal Type**
  - *Single* type that summarises all other types
- Here: use **Parametric Types** with **Type Variables**:
  - $\textsc{List}[\alpha]$ summarises $\textsc{List}[\textsc{Int}]$, $\textsc{List}[\textsc{Bool}]$, $\textsc{List}[\textsc{List}[\ldots]]$
- Difference τ, α when computing type:
  - τ: must solve when we see AST node
  - α: is a valid type already
    - Can make it more concrete later

Typing Rules: Parametric Types

— First attempt —

$$\frac{}{\mathsf{true} : \textsc{Bool}}\ (\textit{t-true}) \qquad \frac{}{\mathsf{false} : \textsc{Bool}}\ (\textit{t-false}) \qquad \frac{v \in \mathit{nat}}{v : \textsc{Int}}\ (\textit{t-nat})$$

$$\frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1\ \mathsf{plus}\ e_2 : \textsc{Int}}\ (\textit{t-plus}) \qquad \frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 >= e_2 : \textsc{Bool}}\ (\textit{t-ge}) \qquad \frac{\Delta(\underline{x}) = \tau}{\underline{x} : \tau}\ (\textit{t-var})$$

$$\frac{e_1 : \textsc{Bool} \quad e_2 : \tau \quad e_3 : \tau}{\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 : \tau}\ (\textit{t-if}) \qquad \frac{e_1 : \tau_1 \quad \Delta(\underline{x}) = \tau_1 \quad e_2 : \tau_2}{\mathsf{let}\ \underline{x} = e_1\ \mathsf{in}\ e_2 : \tau_2}\ (\textit{t-let})$$

$$\frac{}{\mathsf{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\mathsf{cons}(e_1, e_2) : \textsc{List}[\tau]}\ (\textit{t-cons})$$

Originally $\Delta(\underline{x}) = \textsc{List}[\alpha]$

Must merge $\textsc{List}[\alpha] = \textsc{List}[\textsc{Int}]$

Analogous to variable types

$\Delta(\underline{x}) =$ ~~$\textsc{List}[\alpha]$~~ $\textsc{List}[\textsc{Int}]$

$\Delta(\alpha) = \textsc{Int}$

$$\frac{\dfrac{}{\mathsf{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \Delta(\underline{x}) = \textsc{List}[\alpha] \qquad \dfrac{\dfrac{1 \in \mathit{nat}}{1 : \textsc{Int}}\ (\textit{t-nat}) \qquad \dfrac{\Delta(\underline{x}) = \textsc{List}[\textsc{Int}]}{\underline{x} : \textsc{List}[\textsc{Int}]}}{\mathsf{cons}(1, \underline{x}) : \textsc{List}[\textsc{Int}]}}{\mathsf{let}\ \underline{x} = \mathsf{nil}\ \mathsf{in}\ \mathsf{cons}(1, \underline{x}) : \textsc{List}[\textsc{Int}]}\ (\textit{t-let})$$

Typing Rules: Parametric Types

— First attempt —

$$\frac{}{\text{true} : \textsc{Bool}}\ (\textit{t-true}) \qquad \frac{}{\text{false} : \textsc{Bool}}\ (\textit{t-false}) \qquad \frac{v \in \textit{nat}}{v : \textsc{Int}}\ (\textit{t-nat})$$

$$\frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 \text{ plus } e_2 : \textsc{Int}}\ (\textit{t-plus}) \qquad \frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 >= e_2 : \textsc{Bool}}\ (\textit{t-ge}) \qquad \frac{\Delta(\underline{x}) = \tau}{\underline{x} : \tau}\ (\textit{t-var})$$

$$\frac{e_1 : \textsc{Bool} \quad e_2 : \tau \quad e_3 : \tau}{\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 : \tau}\ (\textit{t-if}) \qquad \frac{e_1 : \tau_1 \quad \Delta(\underline{x}) = \tau_1 \quad e_2 : \tau_2}{\text{let } \underline{x} = e_1 \text{ in } e_2 : \tau_2}\ (\textit{t-let})$$

$$\frac{}{\text{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\text{cons}(e_1, e_2) : \textsc{List}[\tau]}\ (\textit{t-cons})$$

Circular type — *t-cons* requires:

$$\begin{aligned} \tau &= \textsc{List}[\alpha] \\ \textsc{List}[\tau] &= \textsc{List}[\alpha] \end{aligned}$$

$\Delta(\alpha) = \textsc{List}[\alpha]$

$$\frac{\dfrac{}{\text{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \dfrac{}{\text{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil})}{\text{cons}(\text{nil}, \text{nil}) : \textbf{?}}\ (\textit{t-cons})$$

Type Variable Freshness

- Our typing rule for nil doesn't work as intended:
  All nil use the same α in their type
  $\implies$ all lists must have the same type

$$\frac{\alpha \quad \textit{fresh}}{\text{nil} : \textsc{List}[\alpha\alpha]} \;(\textit{t-nil}) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\text{cons}(e_1, e_2) : \textsc{List}[\tau]} \;(\textit{t-cons})$$

- Fix: We create a *fresh* type variable for every nil
  (specific to AST node, different from all others)

$\Delta(\alpha_1) = (\textit{not set})$

$\Delta(\alpha_2) = (\textit{not set})\textsc{List}[\alpha_1]$

$\tau \mapsto \textsc{List}[\alpha_1]$

$$\frac{\dfrac{\alpha_1 \quad \textit{fresh}}{\text{nil} : \tau\textsc{List}[\alpha_1]} \;(\textit{t-nil}) \qquad \dfrac{\alpha_2 \quad \textit{fresh}}{\text{nil} : \textsc{List}[\tau\textsc{List}[\alpha_1]]\textsc{List}[\alpha_2]} \;(\textit{t-nil})}{\text{cons}(\text{nil}, \text{nil}) : \textsc{List}[\textsc{List}[\alpha_1]]} \;(\textit{t-cons})$$

Summary

- Some expressions may have an unbounded number of types
- We can usually use **type variables** to present these types compactly
- This produces **principal types** if we can summarise *all* types
- **Parametric types** (or *parametrically polymorphic*) types arise frequently
- Correctly using expressions with type variables may require us to produce **fresh type variables**
- Still need to solve:
  How *do* we merge type variables in equations?

$$\textsc{List}[\alpha_1] = \textsc{List}[\textsc{List}[\alpha_2]]$$

Parametric Types in Practice

- Widely used today, e.g. *Generics* in Java:

Java	**Scala**
`List<E>`	`List[A]`
`Set<E>`	`Set[A]`
`Map<K, V>`	`Map[K, V]`

- Also used as the type of *functions*:

Java	**Scala**	**Common**
`Function<T, R>`	`A => B`	$\alpha \to \beta$

- Scala and others also support parametric *tuple types*:

Scala	**Ocaml/SML**	**Common**
`(A, B, C)`	`'a * 'b * 'c`	$\alpha \times \beta \times \gamma$

- We often combine tuple and function types when inferring types of functions:

$$\texttt{countOccurrencesInList} : \textsc{List}[\alpha] \times \alpha \to \textsc{Int}$$

Typing Challenge 1

- Let's try to do (static) type inference for Python
  (No typing rules, focus on intuition)
- What should the type of f be:

Python

```
def f(x:int) :
    return [x]
```

- f: $\ldots\textsc{Int} \rightarrow \ldots\textsc{List}[\textsc{Int}]$
- x has no initialiser, but x:int makes type clear

Typing Challenge 2

- Let's try to do (static) type inference for Python
  (No typing rules, focus on intuition)
- What should the type of g be:

Python

```
def g(y) :
    return [y]
```

- g: $\ldots \alpha \rightarrow \ldots \textsc{List}[\alpha]$
- No information that helps us find type of y!
- Using type variable captures idea:
  input type determines output type

Solving Typing Challenge 2

Python

```
def g(y) :
    return [y]
```

- Can we apply *t-var* for g's y?

$$\frac{\Delta(x) = \tau}{x : \tau} \ (\textit{t-var})$$

- **No:** Type of y never defined in this code

⇒ no τ exists for $\Delta(x) = \tau$

⇒ Cannot satisfy premise of rule *t-var*

- If we don't know the concrete type, anything is possible
- So let's assign a fresh type variable as type of each variable:

$$\frac{\Delta(x) = \alpha \quad \alpha \ \textit{fresh}}{x : \alpha} \ (\textit{t-var}')$$

Typing Variables with Type Variables

$$\frac{\Delta(\underline{x}) = \alpha \quad \alpha \ \textit{fresh}}{\underline{x} : \alpha} \ (\textit{t-var}')$$

Python

```
x = 1



x = "foo"
```

$\Delta(\underline{x}) = \alpha$
$1 : \textsc{Int}$
$\alpha = \textsc{Int}$

$\Delta(\underline{x}) = \beta$
$\texttt{"foo"} : \textsc{String}$
$\beta = \textsc{String}$

- Each assignment involves variable $\underline{x}$
- Create one type variable per use of $\underline{x}$
- Observe that $\alpha = \textsc{Int}$
- Observe that $\beta = \textsc{String}$
- Note that $\alpha = \Delta(\underline{x}) = \beta$:
  Found type error
- Simplification: attach type variable to *declaration* of $\underline{x}$, so we only need α

Simplified use of Type Variables

$$\frac{\Delta(\underline{x}) = \alpha \quad \alpha \textit{ fresh}}{\underline{x} : \alpha} \ (\textit{t-var}')$$

Python

```
x = 1
```

$\underline{x} : \alpha$

$1 : \textsc{Int}$

$\alpha = \textsc{Int}$

```
x = "foo"
```

$\underline{x} : \alpha$

$\texttt{"foo"} : \textsc{String}$

$\alpha = \textsc{String}$

Simplified Notation

- Only one type variable per variable declaration
- Write $\underline{x} : \alpha$ instead of $\Delta(\underline{x}) = \alpha$
- Observe: two kinds of "constraints" from code:
  - Typings: $\underline{v} : \tau$
  - Equalities: $\tau_1 = \tau_2$

Type Inference with Variables: Example

Python

```
def gen(a:map, b:set):
1   m = {}
2   for v in b:
3       if v in a.keys():
4           x = a[v]
5           m[x] = x
6   return m
```

Extract *typings*:

$y : \tau$

Extract *equality constraints*:

$\tau_1 = \tau_2$

$$
\begin{array}{lrcl}
 & a & : & \texttt{map}[\beta_1,\ \beta_2] \\
 & b & : & \texttt{set}[\gamma] \\
 & gen & : & \texttt{map}[\beta_1,\ \beta_2] \times \texttt{set}[\gamma] \rightarrow \xi \\
1 & m & : & \texttt{map}[\alpha_1,\ \alpha_2] \\
2 & v & : & \gamma \\
3 & v & : & \beta_1 \\
 & \gamma & = & \beta_1 \\
4 & x & : & \alpha_3 \\
 & a & : & \texttt{map}[\gamma,\ \alpha_3] \\
 & \texttt{map}[\beta_1,\ \beta_2] & = & \texttt{map}[\gamma,\ \alpha_3] \\
5 & m & : & \texttt{map}[\alpha_3,\ \alpha_3] \\
 & \texttt{map}[\alpha_1,\ \alpha_2] & = & \texttt{map}[\alpha_3,\ \alpha_3] \\
6 & m & : & \xi \\
 & \xi & = & \texttt{map}[\alpha_1,\ \alpha_2]
\end{array}
$$

How do we solve this automatically?

Summary

- Identifiers may not make their type conveniently accessible:
  - Identifier is "under-constrained" (generic)
  - Type of identifier depends on itself (e.g., identifier is name of a recursive function)
- Side-step by adding indirection: Type variables $\alpha, \beta, \ldots$
- With this approach, analysing code produces two kinds of *constraints*:

1. Typings: $\underline{x} : \tau$
2. Equality constraints: $\tau_1 = \tau_2$

- Completing type analysis requires solving these constraints

Type Inference: Constraints

Typings:

$$
\begin{aligned}
\mathsf{a} &: \texttt{map[}\beta_1\texttt{, }\beta_2\texttt{]} \\
\mathsf{b} &: \texttt{set[}\gamma\texttt{]} \\
\mathsf{gen} &: \texttt{map[}\beta_1\texttt{, }\beta_2\texttt{]} \times \texttt{set[}\gamma\texttt{]} \to \xi \\
\mathsf{m} &: \texttt{map[}\alpha_1\texttt{, }\alpha_2\texttt{]} \\
\mathsf{v} &: \gamma \\
\mathsf{v} &: \beta_1 \\
\mathsf{x} &: \alpha_3 \\
\mathsf{a} &: \texttt{map[}\gamma\texttt{, }\alpha_3\texttt{]} \\
\mathsf{m} &: \texttt{map[}\alpha_3\texttt{, }\alpha_3\texttt{]} \\
\mathsf{m} &: \xi
\end{aligned}
$$

Type Equality Constraints:

$$
\begin{aligned}
\gamma &= \beta_1 \\
\texttt{map[}\beta_1\texttt{, }\beta_2\texttt{]} &= \texttt{map[}\gamma\texttt{, }\alpha_3\texttt{]} \\
\texttt{map[}\alpha_1\texttt{, }\alpha_2\texttt{]} &= \texttt{map[}\alpha_3\texttt{, }\alpha_3\texttt{]} \\
\xi &= \texttt{map[}\alpha_1\texttt{, }\alpha_2\texttt{]}
\end{aligned}
$$

Unification

$$\begin{array}{rcl} \gamma & = & \beta_1 \\ \texttt{map}[\beta_1, \ \beta_2] & = & \texttt{map}[\gamma, \ \alpha_3] \\ \texttt{map}[\alpha_1, \ \alpha_2] & = & \texttt{map}[\alpha_3, \ \alpha_3] \\ \xi & = & \texttt{map}[\alpha_1, \ \alpha_2] \end{array}$$

- *Unification* describes the problem of solving such equations
- Some unification problems are undecidable
  - *Subtyping* in particular usually leads to undecidability
- Our problem has an efficient (near-linear) solution:
  - Given a *worklist* of equality constraints:
  - Remove and process one constraint at a time
  - If constraint has form $\alpha = \tau$: replace $\alpha \mapsto \tau$
  - Otherwise, break equation into smaller equalities, add to worklist
  - ... plus some minor tweaks

First, let us simplify our representation

Type Constructors

- Recall Parametric Types:
  - `Set[`α`]`
  - `Map[`α`, `β`]`
- Type constructors: things like `Set`, `Map`
  - Take type parameters α, β
  - Build new type
- Other type constructors:
  - $\cdots \times \cdots \times \cdots$: constructs product types
  - $\rightarrow$: constructs function types
- General notation: $C_i^k(\tau_1, \ldots, \tau_k)$
  - E.g.: $\texttt{int} \rightarrow \texttt{string} = C^2_{\rightarrow}(C^0_{\texttt{int}}, C^0_{\texttt{string}})$
  - E.g.: $Set[Set[\texttt{int}]] = C^1_{\texttt{Set}}(C^1_{\texttt{Set}}(C^0_{\texttt{int}}))$
- k: arity of type constructor
- i: globally unique identifier for constructor

Type Unification

- Each equation has one of these forms:

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Solution: Replace $\beta \mapsto \alpha$ everywhere
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by equations:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Solution: Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$ everywhere
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots) \Rightarrow$ *Type Error* (*"Occurs Check"*)

(Martelli and Montanari, 1982, based on Robinson, 1965)

Example (Continued)

$$\text{gen} : \texttt{map}[\beta_1,\ \beta_2] \times \texttt{set}[\gamma] \to \xi$$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{array}{rcl} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

     $\Rightarrow$ *Type Error*

$$\begin{array}{rcl} \gamma & = & \beta_1 \\ \texttt{map}[\beta_1,\ \beta_2] & = & \texttt{map}[\gamma,\ \alpha_3] \\ \texttt{map}[\alpha_1,\ \alpha_2] & = & \texttt{map}[\alpha_3,\ \alpha_3] \\ \xi & = & \texttt{map}[\alpha_1,\ \alpha_2] \end{array}$$

Example (Continued)

$$\text{gen} : \texttt{map}[\beta_1, \ \beta_2] \ \times \ \texttt{set}[\beta_1] \rightarrow \ \xi$$

1 $\alpha = \alpha$ (trivial)

2 $\alpha = \beta$

- Replace $\beta \mapsto \alpha$

3 $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$

- *Type Error* if $i \neq j$ or $k \neq l$
- Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4 $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$

- Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
- **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

$\Rightarrow$ *Type Error*

2

$$\begin{array}{rcl} \sout{\gamma} & \sout{=} & \sout{\beta_1} \\ \texttt{map}[\beta_1, \ \beta_2] & = & \texttt{map}[\beta_1, \ \alpha_3] \\ \texttt{map}[\alpha_1, \ \alpha_2] & = & \texttt{map}[\alpha_3, \ \alpha_3] \\ \xi & = & \texttt{map}[\alpha_1, \ \alpha_2] \end{array}$$

Example (Continued)

gen : $\texttt{map}[\beta_1, \beta_2] \times \texttt{set}[\beta_1] \rightarrow \xi$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{aligned} \tau_1^a &= \tau_1^b \\ \ldots & \quad \ldots \\ \tau_k^a &= \tau_k^b \end{aligned}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

   $\Rightarrow$ *Type Error*

2 ~~$\gamma = \beta_1$~~

3 ~~$\texttt{map}[\beta_1, \beta_2] = \texttt{map}[\beta_1, \alpha_3]$~~

$$\begin{aligned} \texttt{map}[\alpha_1, \alpha_2] &= \texttt{map}[\alpha_3, \alpha_3] \\ \xi &= \texttt{map}[\alpha_1, \alpha_2] \\ \beta_1 &= \beta_1 \\ \beta_2 &= \alpha_3 \end{aligned}$$

Example (Continued)

gen : $\texttt{map}[\beta_1, \ \beta_2] \times \texttt{set}[\beta_1] \to \xi$

1 $\alpha = \alpha$ (trivial)

2 $\alpha = \beta$

- Replace $\beta \mapsto \alpha$

3 $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$

- *Type Error* if $i \neq j$ or $k \neq l$
- Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4 $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$

- Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
- **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

$\Rightarrow$ *Type Error*

2 ~~$\gamma = \beta_1$~~

3 ~~$\texttt{map}[\beta_1, \ \beta_2] = \texttt{map}[\beta_1, \ \alpha_3]$~~

3 ~~$\texttt{map}[\alpha_1, \ \alpha_2] = \texttt{map}[\alpha_3, \ \alpha_3]$~~

$$\begin{array}{rcl} \xi & = & \texttt{map}[\alpha_1, \ \alpha_2] \\ \beta_1 & = & \beta_1 \\ \beta_2 & = & \alpha_3 \\ \alpha_1 & = & \alpha_3 \\ \alpha_2 & = & \alpha_3 \end{array}$$

Example (Continued)

gen : $\texttt{map}[\beta_1, \beta_2] \times \texttt{set}[\beta_1] \rightarrow \texttt{map}[\alpha_1, \alpha_2]$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{aligned} \tau_1^a &= \tau_1^b \\ \ldots & \quad \ldots \\ \tau_k^a &= \tau_k^b \end{aligned}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

   $\Rightarrow$ *Type Error*

2	~~γ~~	~~$=$~~	~~β_1~~
3	~~$\texttt{map}[\beta_1, \beta_2]$~~	~~$=$~~	~~$\texttt{map}[\beta_1, \alpha_3]$~~
3	~~$\texttt{map}[\alpha_1, \alpha_2]$~~	~~$=$~~	~~$\texttt{map}[\alpha_3, \alpha_3]$~~
4	~~ξ~~	~~$=$~~	~~$\texttt{map}[\alpha_1, \alpha_2]$~~
	β_1	$=$	β_1
	β_2	$=$	α_3
	α_1	$=$	α_3
	α_2	$=$	α_3

Example (Continued)

gen : $\texttt{map}[\beta_1, \beta_2] \times \texttt{set}[\beta_1] \rightarrow \texttt{map}[\alpha_1, \alpha_2]$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

   $\Rightarrow$ *Type Error*

2	~~γ~~	~~$=$~~	~~β_1~~
3	~~$\texttt{map}[\beta_1, \beta_2]$~~	~~$=$~~	~~$\texttt{map}[\beta_1, \alpha_3]$~~
3	~~$\texttt{map}[\alpha_1, \alpha_2]$~~	~~$=$~~	~~$\texttt{map}[\alpha_3, \alpha_3]$~~
4	~~ξ~~	~~$=$~~	~~$\texttt{map}[\alpha_1, \alpha_2]$~~
1	~~β_1~~	~~$=$~~	~~β_1~~
	β_2	$=$	α_3
	α_1	$=$	α_3
	α_2	$=$	α_3

Example (Continued)

gen : $\texttt{map}[\beta_1, \beta_2] \times \texttt{set}[\beta_1] \rightarrow \texttt{map}[\alpha_1, \alpha_2]$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$
     $\Rightarrow$ *Type Error*

2	~~γ~~	~~$=$~~	~~β_1~~
3	~~$\texttt{map}[\beta_1, \beta_2]$~~	~~$=$~~	~~$\texttt{map}[\beta_1, \alpha_3]$~~
3	~~$\texttt{map}[\alpha_1, \alpha_2]$~~	~~$=$~~	~~$\texttt{map}[\alpha_3, \alpha_3]$~~
4	~~ξ~~	~~$=$~~	~~$\texttt{map}[\alpha_1, \alpha_2]$~~
1	~~β_1~~	~~$=$~~	~~β_1~~
2	~~β_2~~	~~$=$~~	~~α_3~~
	α_1	$=$	β_2
	α_2	$=$	β_2

Example (Continued)

gen : $\texttt{map}[\beta_1,\ \beta_2] \times \texttt{set}[\beta_1] \rightarrow \texttt{map}[\beta_2,\ \alpha_2]$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

   $\Rightarrow$ *Type Error*

2	~~γ~~	~~$=$~~	~~β_1~~
3	~~$\texttt{map}[\beta_1,\ \beta_2]$~~	~~$=$~~	~~$\texttt{map}[\beta_1,\ \alpha_3]$~~
3	~~$\texttt{map}[\alpha_1,\ \alpha_2]$~~	~~$=$~~	~~$\texttt{map}[\alpha_3,\ \alpha_3]$~~
4	~~ξ~~	~~$=$~~	~~$\texttt{map}[\alpha_1,\ \alpha_2]$~~
1	~~β_1~~	~~$=$~~	~~β_1~~
2	~~β_2~~	~~$=$~~	~~α_3~~
2	~~α_1~~	~~$=$~~	~~β_2~~
	α_2	$=$	β_2

Example (Continued)

$\text{gen} : \texttt{map}[\beta_1,\ \beta_2] \times \texttt{set}[\beta_1] \rightarrow \texttt{map}[\beta_2,\ \beta_2]$

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace $\beta \mapsto \alpha$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - *Type Error* if $i \neq j$ or $k \neq l$
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$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $\alpha \mapsto C_i^k(\tau_1, \ldots, \tau_k)$
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$

     $\Rightarrow$ *Type Error*

- 2 ~~$\gamma = \beta_1$~~
- 3 ~~$\texttt{map}[\beta_1,\ \beta_2] = \texttt{map}[\beta_1,\ \alpha_3]$~~
- 3 ~~$\texttt{map}[\alpha_1,\ \alpha_2] = \texttt{map}[\alpha_3,\ \alpha_3]$~~
- 4 ~~$\xi = \texttt{map}[\alpha_1,\ \alpha_2]$~~
- 1 ~~$\beta_1 = \beta_1$~~
- 2 ~~$\beta_2 = \alpha_3$~~
- 2 ~~$\alpha_1 = \beta_2$~~
- 1 ~~$\alpha_2 = \beta_2$~~

Substituting "Everywhere"?

- The Martelli/Montanari algorithm asks us to "replace type variables everywhere":
  - "Solution: Replace β with α everywhere"
  - "Solution: Replace $C_i^k(\tau_1, \ldots, \tau_k)$ for α everywhere"
- Implementation strategies?:
  - **Substitute systematically**:
    - Replace everywhere in worklist
    - Replace everywhere in solutions (e.g., symbol table)
  - **Update Lists**:
    - 'Substitute systematically', but on demand, storing pending updates
  - **Stateful type variables**: (*my recommendation*)
    - Type variables remember their bindings, e.g. in $\Delta(\alpha)$
    - Some challenges with nontrivial merges

Summary

- During type analysis, we often encounter nontrivial equations over types
- To check these and extract relevant equalities, we use **Unification**
- The **Martelli/Montanari algorithm** is efficient for the types we have discussed so far
- Input:
  - A list of equations over types
- Output:
  - Bindings to type variables
  - Type variables such as α may be:
    - Replaced by a concrete type, such as $\textsc{Int}$
    - Replaced by another type variable, such as β
    - Replaced by a partially abstract type, such as $\textsc{List}[\gamma]$

Merging Variables

- Consider solving:

$$\begin{aligned} \alpha &= \beta \\ \beta &= \gamma \\ \gamma &= \delta \\ \delta &= \xi \end{aligned}$$

- Implementing unification with stateful variables naively can make it costly to figure out the "real" type of α:

$\Delta(\alpha) = \beta$
$\Delta(\beta) = \gamma$
$\Delta(\gamma) = \delta$
$\Delta(\delta) = \xi$

- Fast unification implementations instead use UNION-FIND datastructures

Union-Find Datastructures

Java

```java
public class UFSet {
  UFSet repr = null;

  // Find & update representative
  public UFSet find() {
    UFSet r = this;
    while (r.repr != null) {
      r = r.repr;
    }
    this.repr = r;
    return r;
  }

  public void union(UFSet other) {
    other = other.find();
    UFSet r = this.find();
    // we can update r or other
    if (r != other) {
      other.repr = r;
  } }

  public boolean equals(UFSet o) {
    return this.find() == o.find();
} }
```

Summary

- UNION-FIND datastructure can speed up type variable merging
- Type variables represent a set of equivalent variables
- Each set has one representative
- *find* operation finds that representative
  - updates cached references to it
- *union*(v_1, v_2) operation finds representatives r_1, r_2 of two variables
  - If $r_1 \neq r_2$, v_1, v_2 in different set
  - Then, update either representative of v_1 to now be v_2, or vice-versa
  - High-performance implementations make this decision based on:
    - set size
    - estimated "depth" of representative chains ('*rank*')

Unification, Types, and Re-use

Teal-0

```
fun id(x:α1):α2 = return x:α1;

var b:β1 := id("foo":string):β2
var c:γ1 := id(15:int):γ2
```

- What are the types here?
- We have $\alpha_1 = \alpha_2 = \texttt{string} = \texttt{int}$: type error!

Approach doesn't allow `id` on both `string` and `int`

Type Schemes

- We had the same issue before:

$$\frac{}{\text{nil} : \textsc{List}[\alpha]} \quad\Longrightarrow\quad \frac{\alpha \quad \textit{fresh}}{\text{nil} : \textsc{List}[\alpha]}$$

- We want a similar scheme for `id`: *create fresh type variables*
- However, we can't write custom rules for all user-defined functions!
- Polymorphism with user-defined functions:
  - *Type Schemes* (or *Polytypes*):

(1)	"normal" polymorphic type:	$\alpha \to \alpha$
(2)	variables to replace by fresh ones:	$\{ \alpha \}$
	short notation for (1)+(2):	$\forall\alpha.\alpha \to \alpha$

  - `id` : $\forall\alpha.\alpha \to \alpha$
  - *Instantiate* type schemes with fresh type variables on demand:

```
        id: α2 → α2
var b := id("foo");
```

```
        id: α3 → α3
var b := id(255);
```

Using Type Schemes

- If we have a type scheme: *instantiate* scheme to use it
- Instantiating type schemes: (formalises of the last slide):

$$\frac{\Delta(\underline{x}) = \forall \alpha_1, \ldots, \alpha_n . \tau \quad \beta_i \textit{ fresh}, i \in \{1, \ldots, n\}}{\underline{x} : \tau[\alpha_1 \mapsto \beta_1, \ldots, \alpha_n \mapsto \beta_n]} \; (\textit{t-var-inst})$$

- If we *want* a type scheme: *abstract* type into type scheme
- Abstracting type schemes:

1. Infer type via unification: $\underline{f} : \tau$
2. Figure out which set of type variables to abstract: $\mathcal{T}$
3. **Update** type schema: $\Delta(\underline{f}) = \forall \mathcal{T} . \tau$

How do we find $\mathcal{T}$?

Summary

- Represent polymorphic types as type **Schemes**
- Abstract over free type variables ($\forall$) to introduce schemes
- *Instantiate* schemes into types when referenced

Finding Type Schemes (1/3)

Teal-0

```
fun id(x) = return x;

var b = id("foo");
var c = id(17);
```

- When should we build the schema for `id`?
  - **After all unification:**
    Too late: have already run into type error (cf. earlier)
  - **Before all unification:**
    Too early:
    - Our first type constraint was: `id` : $\alpha_1 \rightarrow \alpha_2$
    - However, `id` : $\forall \alpha_1, \alpha_2 . \alpha_1 \rightarrow \alpha_2$ would be wrong:
      Tells us nothing about connection between α_1 and α_2
  - **During Unification:**
    - Must abstract *after* unifying all variables that "matter" to `id`
    - Must abstract *before* we use `id`'s type

Finding Type Schemes (2/3)

Teal-0

```
fun f(x0, x1, x2) = return g(x0 - 1, x1, x2);

fun g(y0, y1, y2) =
    if y0 == y1 {
        return y1;
    } else {
        return f(y1, y0, y2);
    }
```

- Can't build schema of `f` without analysing `g`
- Can't build schema of `g` without analysing `f`

***Mutual dependency*: Can't fully analyse one before the other**

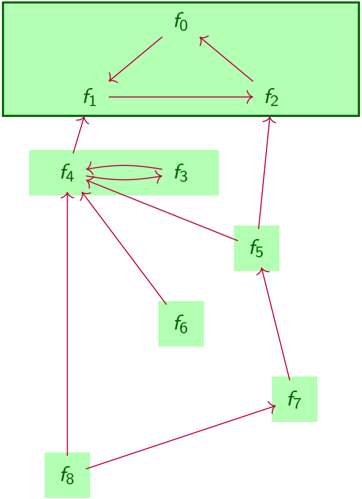
Finding Type Schemes (3/3)

- When functions call each other: must analyse them together
- Generalises to indirect calls
- Find *dependencies*:
  - if `f` calls `g`:
    - `f` *depends* (*directly*) on `g`
  - if `f` *depends* on `g` and `g` *depends* on `h`:
    - `f` *depends* on `h`
- `f` *depends* on `g`: Can't build schema for `f` before analysing `g`

$\Rightarrow$ Analyse such `f` and `g` together

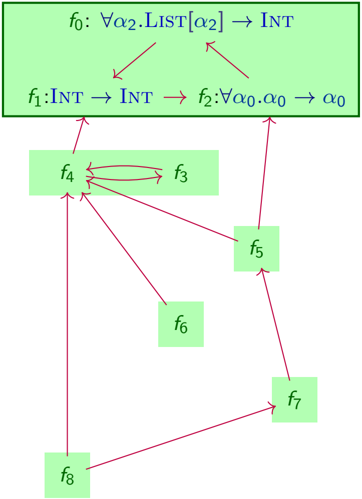
Polymorphic Type Inference Procedure

1. Determine dependencies
2. Cluster mutual dependencies
3. Mark clusters: untyped
4. While there are untyped clusters:
    5. Pick cluster that has no untyped dependencies



Polymorphic Type Inference Procedure

1. Determine dependencies
2. Cluster mutual dependencies
3. Mark clusters: untyped
4. While there are untyped clusters:
   5. Pick cluster that has no untyped dependencies
   6. Analyse all definitions in cluster:
      - Create fresh type variables as needed
      - Record typings: $\texttt{x} : \alpha$
      - Collect type equalities: $\tau_a = \tau_b$
   7. Run Unification
   8. For all definitions $\texttt{f} : \tau$ in cluster:
      - Let $\mathcal{T}$ = all type variables in τ
      - Set $\texttt{f} : \forall\mathcal{T}.\tau$



Summary

- **Polymorphic Type Inference**: generalise types with **Schemes**
- Algorithm:
  - Introduce type variables
  - Systematically apply typing rules to:
    - Generate typings
    - Generate type equality constraints
  - Unify equality constraints 'at the right time'
  - Abstract over free type variables ($\forall$) to introduce schemes
- Must analyse **Dependencies** between definitions
- Unify / abstract when:
  - Finished all dependencies
- Limitations:
  - Does not handle "inner functions"
    (See Damas-Hindley-Milner, Algorithms $\mathcal{W}$ / $\mathcal{J}$ if interested)
  - Type schemes over mutable objects (arrays etc.) unsound
  - Does not handle subtypes
    - Adding subtypes: see Dolan & Mycroft's "MLsub", 2017

Outlook

- **Remember**:
  - Exercise 1 will be released today
  - Exercise 0 still has lab priority for presenting this week
- Next Week:
  - Data Flow Analysis

`http://cs.lth.se/EDAP15`