EDAA40 Exam

4 June 2022

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the *least favourable assumption* about what you intended to write.

A sheet with common symbols and notations is attached at the end.

Good luck!

1	2	3	4	5	**total**
20	24	20	24	12	**100**

Total points: 100
points required for 3: 50
points required for 4: 67
points required for 5: 85

1 [20 p]

1. For $k \in \mathbb{N}^+$, define the set D_k of positive natural numbers strictly larger than k that are divisible by k.

$$D_k = \{n \in \mathbb{N}^+ : (n > k) \wedge (k \mid n)\}$$

2. Define the set $\mathbb{P}$ of prime numbers. (You can use the sets D_k.)

$$\mathbb{P} = \{\mathbb{N}^+ \setminus (\{1\} \cup \bigcup_{k \in \mathbb{N}^+, k \geq 2} D_k)\}$$

or $\{n \in \mathbb{N}^+ : ((q \in \mathbb{N}^+) \wedge (q \mid n)) \rightarrow ((q = 1) \vee (q = n))\}$ or ...

3. *Recursively* define a function $f : \mathbb{N}^+ \longrightarrow \mathbb{P}$ that given a positive natural number $n \in \mathbb{N}^+$ returns the nth prime number. For example, $f(1) = 2$, $f(2) = 3$ and $f(5) = 11$.

For our purposes here, you can use the min function over any subset of $\mathbb{N}$. For example, if D_2 is the set of all positive natural numbers strictly larger than 2 that are divisible by 2, then

$$\min D_2 = 4\,.$$

You can use a helper function if you like.

$$f : n \mapsto \begin{cases} 2 & \text{if } n = 1 \\ \min(\mathbb{P} \setminus \{n \in \mathbb{N}^+ : n \leq f(n-1)\}) & \text{if } n > 1 \end{cases}$$

Alternatively, we could have defined a helper function $f' : \mathbb{N}^+ \times \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{N}$ that given $n \in \mathbb{N}^+$ and $S \subseteq \mathbb{N}$ returns the nth smallest element in S:

$$f' : n, S \mapsto \begin{cases} \min S & \text{if } n = 1 \\ f'(n-1, S \setminus \{\min S\}) & \text{if } n > 1 \end{cases}$$

We would then have

$$f(n) = f'(n, \mathbb{P}).$$

4. Is f bijective? Is it injective? Is it surjective?

f is bijective and therefore also injective and surjective.

2 [24 p]

Let (V, E) be a directed graph. For any vertex $v \in V$, we write $E^0(v) = \{v\}$ and for $n > 0$, $E^n(v) = E(E^{n-1}(v))$.

Recall that a path is a finite sequence $v_0 v_1 \ldots v_k$ of vertices $v_i \in V$ such that for all $i \in \{1, 2, \ldots, k\}$ it is the case that $(v_{i-1}, v_i) \in E$. We say the path $v_0 v_1 \ldots v_k$ has length k (that is, the number of edges in the path, or one less than the number of vertices it contains). We only consider paths with at least one vertex. A path of length 0 is just a vertex itself. Also, a cycle is a path length at least 1 that begins and ends with the same vertex.

1. Write a formula that is true if and only if the graph (V, E) contains at least one cycle.

   $\exists v \in V \exists n \in \mathbb{N}^+ (v \in E^n(v))$ or $\exists v \in V (v \in E^+(v))$ etc.

2. Assume that (V, E) has at least one cycle. Using the definition above, give a formula for the length of the shortest cycle in (V, E).

   $\min\{n \in \mathbb{N}^+ : \exists v \in V (v \in E^n(v))\}$

3. Define a relation $R \subseteq V \times V$ such that for two vertices $v, w \in V$, it is the case that $(v, w) \in R$ if and only if there is a path from v to w and there is a path from w to v.

   $R = \{(v, w) \in V^2 : \exists m, n \in \mathbb{N} (v \in E^n(w) \land w \in E^m(v))\}$
   or, e.g., $\{(v, w) \in V^2 : v \in E[w]) \land w \in E[v]\}$

4. Is R transitive? Is it symmetric? Is it reflexive?

   R is transitive (always). R is symmetric (always). R is reflexive (always).

5. Define a relation $S \subseteq V \times V$ such that for two vertices $v, w \in V$, it is the case that $(v, w) \in S$ if and only if there is a path from v to w, there is a path from w to v, and the shortest path from v to w is longer than the shortest path from w to v.

   $S = \{(v, w) \in R : \exists n \in \mathbb{N} (v \in E^n(w) \land \forall m \in \mathbb{N} (w \in E^m(v) \rightarrow m > n))\}$

6. Is S transitive? Is it symmetric? Is it reflexive?

   S is sometimes transitive, but only in special cases (e.g., if it is empty). S is only symmetric if it is empty. R is only reflexive if it is empty.

3 [20 p]

All *intervals* below are supposed to be intervals in the real numbers, $\mathbb{R}$.

Suppose $f : A \longrightarrow B$ is the function defined by

$$f : x \mapsto x^2$$

1. If $A = \{0, 1, 2, 3\}$, what is B so that f is a bijection?

   $B = \{0, 1, 4, 9\}$

2. If $A = [-2, 3]$, what is B so that f is surjective?

   $B = [0, 9]$

3. If $A = [2, 4]$, what is $f[A]$?

   If $A = [2, 4]$, then $f[A]$ is not well-defined since we have that $f : A \longrightarrow B$.
   The question should have been: "If $A = B = \mathbb{R}^+$, what is $f[[2, 4]]$?"
   In which case the answer would have been: $f[[2, 4]] = [2, \infty[$

4. If $B = [0, 10]$, give an A so that f is injective. For your choice of A, is f^{-1} a function?

   There are two types of possible answers here.
   Option 1: $A = [0, \sqrt{10}]$. In this case f^{-1} is a function.
   It could also be $A = [-\sqrt{10}, 0]$ or other combinations of these intervals.
   Option 2: $A = [0, 3]$. In this case f^{-1} is NOT a function.
   Again, there are several possible choices of A here.

5. If $A = [0, 3]$ and $B = [0, 9]$, what is $f^{-1}\left[\left]0, \frac{1}{3}\right]\right]$?

   The definition in the slides says we can only take closure of functions that have the same domain and codomain. So it would be okay to state this.
   Again, the question should have been "If $A = B = \mathbb{R}^+$, what is $f^{-1}\left[\left]0, \frac{1}{3}\right]\right]$?"
   In which case the answer would have been: $f^{-1}\left[\left]0, \frac{1}{3}\right]\right] =]0, 1[$

4 [24 p]

Let $G = (V, E)$ be a directed graph.

Recall that a path is a finite sequence $v_0 v_1 \ldots v_k$ of vertices $v_i \in V$ such that for all $i \in \{1, 2, \ldots, k\}$ it is the case that $(v_{i-1}, v_i) \in E$. We say the path $v_0 v_1 \ldots v_k$ has length k (that is, the number of edges in the path, or one less than the number of vertices it contains). We only consider paths with at least one vertex. A path of length 0 is just a vertex itself (also called an *empty path*). Also, a cycle is a path length at least 1 that begins and ends with the same vertex.

In a certain strategy game defined on G, there are k players. A directed edge $(u, v) \in E$ is viewed as a one-way road from u to v. We say that it is possible to reach a vertex u from vertex a v if there is a (possibly empty) directed path that starts with v and ends with u.

Each vertex can be owned by at most one player. If a player owns a vertex $v \in V$, then he is allowed to transit toll-free on all roads leaving v and on all road arriving at v. If a player wants to transit on an edge $(u, v) \in E$, and the player does not own neither u nor v, then the player must pay a toll to the players (if any) that own vertices u and v.

Part A. [12 pt]

1. Define the set S_v of vertices that can be reached from the vertex v.

$$S_v = E[v] \text{ or, e.g., } E^+(v) \cup \{v\}$$

2. Let V_i be the set of vertices owned by player i. Given a vertex $v \in V_i$, define the set $F_v(i)$ of all vertices that player i can reach from v without paying any tolls.

$$F_v(i) = (E \cap (V \times V_i \cup V_i \times V))[v]$$

3. Write a formula that is true iff there exists a player that can transit on all roads without paying tolls.

$\exists i \in [k](E \subseteq (V \times V_i \cup V_i \times))$ or, e.g., $\exists i \in [k](\forall (u, v) \in E(u \in V_i \vee v \in V_i))$

Part B. [12 pt] *Recursively* define a function $f : \mathcal{P}(V) \longrightarrow \mathbb{N}$ that given $U \subseteq V$ outputs the minimum number of vertices that a player has to own so that he can travel toll-free on all roads in $U \times U$. Explain in words why your chosen definition is well-defined (i.e., why we don't get stuck in cycles).

$$f : U \mapsto \begin{cases} 0 & \text{if } (U \times U) \cap E = \emptyset \\ \min_{u \in U} f(U \setminus \{u\}) + 1 & \text{otherwise} \end{cases}$$

5 [12 p]

Let p, q, r be Boolean propositional variables. For each of the items below, answer whether the two statements are equivalent. If yes, explain how you got to this conclusion (either explain in words your reasoning or show the steps you took to get the answer). If no, give an example of values of p, q, r for which they differ.

1. $(p \to q) \to r$ and $\neg p \vee \neg q \vee r$

   Not equivalent. For $p = q =$ true and $r =$ false, the first formula evaluate to true, but the second one to false.

2. $\neg((p \vee q) \leftrightarrow (r \wedge q))$ and $(p \wedge \neg q) \vee (q \wedge \neg r)$

   Equivalent. You can justify this by writing out the truth table, or the steps you took to obtain a DNF, or in words.

q	p	r	$(p \vee q)$	$(r \wedge q)$	$\neg((p \vee q) \leftrightarrow (r \wedge q))$	$(p \wedge \neg q)$	$(q \wedge \neg r)$	$(p \wedge \neg q) \vee (q \wedge \neg r)$
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	1	0	1	1	0	1
0	1	1	1	0	1	1	0	1
1	0	0	1	0	1	0	1	1
1	0	1	1	1	0	0	0	0
1	1	0	1	0	1	0	1	1
1	1	1	1	1	0	0	0	0

3. $(p \leftrightarrow q) \to (q \wedge r)$ and $(\neg p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$

   Not equivalent. For $p = q =$ true and $r =$ false, the first formula evaluate to false, but the second one to true.

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e., including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e., they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e., $\exists k \in \mathbb{N}$ such that $ka = b$
- $\mathcal{P}$ power set of A
- $\overline{R}$ of a relation R: its complement
- R^{-1} of a relation R: its inverse
- $R \circ S, f \circ g$ of relations and functions: their composition
- $R[X], f[X]$ closure of a set X under a relation R. a set of relations R or a function f
- $[a,b],]a,b[,]a,b], [a,b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are equinumerous
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements of S
- $\bigcap S$ intersection of all elements of S
- $\bigcup_{a \in S} E(a), \bigcap_{a \in S} E(a)$ generalized union / intersection of the sets computed for every a in S