Combining Loop-shaping and Laguerre Development for Rational Approximation of a Noninteger Order, Integral, Analog Controller

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Abstract

Fractional order systems are receiving more and more attention from the scientific literature as many real systems showed behavior which can be represented by fractional order differential equations. Besides many control design methods are being investigated since fractional order controllers guarantee robust stability in wide frequency ranges. This paper uses generalized Laguerre orthogonal functions to approximate a fractional integrator as a rational transfer function. A truncated Laguerre series is derived in the time domain to represent the approximated impulse response of the irrational transfer function. Then, a loop shaping is performed on the phase plot of the frequency response. Simulation results show how the compact and economic approximation gives good performance in a sufficiently wide frequency range.

1. Introduction

Fractional calculus is 300 years old, but the formalization of the theory is due to Abel (1823), Liouville (1832), Heaviside (1892) and Riemann (1953). However, applying fractional calculus to science and engineering is only a recent research issue. A short survey of researches in viscoelasticity, electrical and electronic circuit design, electroanalytical chemistry, biology, etc., is provided by [9]. Also in control theory and applications, the interest for non integer derivatives and integrals is relatively new. F.e., noninteger order controllers (NOC) proposed by [6-7,10-11] are recent applications.

The NOC are based on the operators s^{-v} (s^{v}), with $0<v<1$, which represent a generalization of common integral (derivative) integer operators in the s-domain. However, the irrational transfer functions s^{-v} (s^{v}) are infinite dimension operators. They can be approximated by rational functions, which generally are high-order.

Today, however, realization of high-order transfer functions is not a problem since analog compensators of order 20 or more are common in analog telecommunication systems [5]. They allow the design of control systems which are robust when the stability margins are prescribed by appropriate continuous boundaries rather than with values of phase/gain margins at a given frequency. This motivates fractional order controllers applications [6-7,10-11].

This paper shows that the generalized Laguerre orthogonal functions allow an easy and economic approximation of irrational functions of type s^{-v} as ratio of polynomials.

Combining the Laguerre series development approach and the loop shaping technique can effectively reduce the order of the approximating transfer function of s^{-v}. The approach is completely new, for two main reasons.

First, the truncated Laguerre series is obtained, by effectively using the time domain behavior corresponding to the irrational transfer function to be approximated.

Second, the shaping is made on the phase response. More precisely, the phase plot of the transform of the truncated Laguerre series is shaped by a simple lag-lead compensator to match the frequency characteristics of the irrational function, with reduced order rational functions.

2. The time domain orthonormal approximation of a noninteger order integrator

Many methods for implementing NOCs approximate their irrational continuous transfer function by models in the form of ratio two polynomials in the complex variable s. F.e., Charef *et al.* used a recurrent structure of pole-zero pairs, called "singularities", on the negative real axis to represent fractional power of binomial factors containing the poles [2].

The approximation of Bode magnitude plots with slope of $-20v$ dB/dec (where v is noninteger) with zig-zag lines is obtained by properly choosing the distribution of the pole-zero pairs. Oustaloup developed analogous ideas for implementing derivative actions [6-7].

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To the best of the author's knowledge, there are no techniques using the time domain data and behavior of NOC for synthesizing approximate transfer function in terms of ratio of two polynomials. The potentialities of this approach have not been experimented yet. Hence, this paper is an attempt in this direction. So, let $g_I(s) = \Gamma(v)\ (s+a)^{-v}$ be a noninteger transfer function, with v and $a > 0$. If a is small enough, $g_I(s) \approx \Gamma(v)\ s^{-v}$ approximates a noninteger integrator. On the other hand, $g_I(x) = \mathcal{L}^{-1}\{g_I(s)\} = x^{v-1}\ e^{-ax}$ is the inverse Laplace ($\mathcal{L}$) transform, and x is the time variable. This simple expression is the starting point of the following procedure.

The first step is to develop $g_I(x)$ into an infinite expansion of orthogonal functions. Even if many choices can be made, the generalized Laguerre orthogonal functions are the more appropriate orthogonal basis for effectively expressing $g_I(x)$ in terms of a truncated series.

Namely, analytical and computer studies have shown that Laguerre functions provide optimal results in approximating functions with exponential dynamics [3]. Such functions, indeed, lead to a simplified calculation of the coefficients of the development. In the case of the exponential dynamics of $g_I(x)$, owing to built-in exponential term of the generalized Laguerre functions, the coefficients of the series assume an analytical closed form.

Moreover, the compression of the information produced by the Laguerre development leads to an accurate approximation of $g_I(x)$ which needs few terms of the series.

On the other hand, restricting the development to few terms is requested not only by the economy of the approximation, but also by inevitable round off errors in the numerical computation. Due to these errors, attempts to get ever closer approximation by increasing the number of terms can lead to poorer results.

Finally, under a reasonable assumption, the Laplace transform of the approximating expression gives the transfer function as ratio of two polynomials in the complex variable s.

Let the generalized Laguerre polynomials be given by [1,14]:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \tag{1}$$

with $n = 0,1,2,\ldots$ and $x \in \Re^+$. The arbitrary real $\alpha > -1$ is the order of generalization. If $\alpha = 0$ the polynomials reduce to the Laguerre polynomials [1,14].

The $L_n^{(\alpha)}(x)$ are orthogonal with respect to the weight function $w^{(\alpha)}(x) = x^{\alpha}\ e^{-x}$, i.e.:

$$\int_0^{\infty} w^{(\alpha)}(x)\ L_k^{(\alpha)}(x)\ L_n^{(\alpha)}(x)\ dx = \delta_{kn}\ \frac{\Gamma(n+\alpha+1)}{n!} \tag{2}$$

where δ_{kn} is the Kronecker delta and $\Gamma(\cdot)$ is the gamma function.

According to the fundamental property of the Laguerre polynomials, provided that

$$\int_0^{\infty} x^{\alpha}\ e^{-x}\ g_I^2(x)\ dx \tag{3}$$

is finite, the series

$$\sum_{n=0}^{\infty} c_n\ L_n^{(\alpha)}(x)\,, \tag{4}$$

with coefficients c_n given by

$$c_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^{\infty} x^{\alpha}\ e^{-x}\ g_I(x)\ L_n^{(\alpha)}(x)\ dx\,, \tag{5}$$

converges to $g_I(x)$ at every continuity point of $g_I(x)$.

Now, to verify the convergence of the Laguerre approximation, the integral (3) must be preliminarily evaluated. With the given $g_I(x)$, the integral (3) becomes:

$$\int_0^{\infty} x^{\eta-1}\ e^{-\mu x}\ dx \tag{6}$$

with $\mu = 1+2a > 0$ and $\eta = \alpha+2v-1$. For the integral (6) to be finite, it must hold $\eta > 0$, i.e. $v > (1-\alpha)/2$ [4].

The polynomials $L_n^{(\alpha)}(x)$ and the weight function $w^{(\alpha)}(x)$ can be employed to derive functions on $\Re^+$ which are orthonormal with respect to a unity weight function. By multiplying the generalized Laguerre polynomials by the square root of their weight function, changing the scale in the x-axis according to $x = \sigma t$, with $\sigma > 0$, and performing a normalization [13], the definition of the generalized Laguerre functions follows:

$$\lambda_n^{(\alpha)}(\sigma,t) = \sqrt{\frac{\sigma\, n!}{\Gamma(n+\alpha+1)}} \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{(\sigma t)^{k+\alpha/2}}{k!} e^{-\sigma t/2}\,. \tag{7}$$

With reference to the expression of $g_I(t)$, the generic coefficient of the series expansion in the Laguerre generalized function is [12]:

$$\begin{aligned} \hat{c}_n &= < g_I(t), \lambda_n^{(\alpha)}(\sigma,t) > = \\ &= \sqrt{\frac{n!}{\sigma\,\Gamma(n+\alpha+1)}} \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{1}{k!} \int_0^{\infty} \tau^{\hat{\eta}-1}\ e^{-\hat{\mu}\tau}\ d\tau \end{aligned} \tag{8}$$

with $\tau = \sigma t$, $\hat{\eta} = k+v+\alpha/2$ and $\hat{\mu} = 0.5+a/\sigma$, and where $< g_I(t),\ \lambda_n^{(\alpha)}(\sigma,t) >$ denotes the inner product of the two functions.

Since the integral on the right side is of the same type of (6), it follows:

$$\hat{c}_n = \sqrt{\frac{n!}{\sigma\,\Gamma(n+\alpha+1)}}\ \sum_{k=0}^{n}(-1)^k \binom{n+\alpha}{n-k}\frac{1}{k!}\frac{\Gamma(\hat{\eta})}{\hat{\mu}^{\hat{\eta}}}\,. \quad (9)$$

The closed form of the coefficient allows the following simple evaluation of the orthonormal approximation:

$$g_{IN}(t) = \sum_{n=0}^{N-1}\hat{c}_n\ \lambda_n^{(\alpha)}(\sigma,t)\,. \quad (10)$$

3. Shaping of the frequency domain plots of the approximation

The Laplace domain approximation corresponding to (10) is obtained, taking into account the transforms of the generalized Laguerre functions [13]:

$$\Lambda_n^{(\alpha)}(\sigma,s) = \beta\left(\frac{1}{s+0.5\sigma}\right)^{1+0.5\alpha}\sum_{k=0}^{n}M_{k,n}^{\alpha}\left(\frac{\sigma}{s+0.5\sigma}\right)^k \quad (11)$$

with $\beta = \sigma^{0.5(1+\alpha)}$ and:

$$M_{k,n}^{\alpha} = (-1)^k\binom{n}{k}\frac{\Gamma(k+0.5\alpha+1)}{\Gamma(k+\alpha+1)}\sqrt{\frac{\Gamma(n+\alpha+1)}{n!}} \quad (12)$$

for $k \le n$ and $M_{k,n}{}^{\alpha} = 0$ for $k > n$.

For even values of α, the $\Lambda_n{}^{(\alpha)}(\sigma,s)$ are rational functions of s that can be generated as the impulse responses of the network of figure 1, where, for each value of n, one should only consider the blocks associated with $M_{k,n}{}^{\alpha}$, for $k = 0$ to n. E.g., for $n = 2$, only $M_{0,2}{}^{\alpha}$, $M_{1,2}{}^{\alpha}$, $M_{2,2}{}^{\alpha}$ and two blocks with transfer function $\sigma/(s+0.5\sigma)$ are included in the network diagram.

However, also if $\alpha << 1$, the exponent in (11) satisfies $1+0.5\alpha \approx 1$ so that the $\Lambda_n{}^{(\alpha)}(\sigma,s)$ can be approximately assumed as rational functions. The remaining factors in (11) are still considered as α-dependent.

A frequency domain discussion illustrates the correctness of the above assumption and the efficiency of the proposed approach. In particular, putting $s = j\omega$, the approximation

$$g_{IN}(j\omega) = \sum_{n=0}^{N-1}\hat{c}_n\ \Lambda_n^{(\alpha)}(\sigma, j\omega) \quad (13)$$

to $g_I(j\omega)$ is analyzed.

The value of α is set to $\alpha = 0.1$, so that $1+0.5\alpha \approx 1$ is assumed to calculate the functions $\Lambda_n{}^{(\alpha)}(\sigma,s)$. Moreover, since in the case of a truncated Laguerre series, σ determines the rate to decay of the generalized Laguerre functions, the choice of this free design parameter can be used to obtain a compact series expansion of $g_I(j\omega)$.

The technical literature offers many procedures [8,13,15] to fix the scale factor σ. Here, an empirical search has been performed which has lead to a 'good' value $\sigma = 0.81$. A deeper search is not necessary, since more accuracy in the resulting approximation is left to the successive shaping step.

With this choice, the first $N = 8$ coefficients corresponding to $\nu = 0.5$ are:

$$\hat{c}_0 = 2.1798 \quad \hat{c}_1 = 0.2977 \quad \hat{c}_2 = 0.6664 \quad \hat{c}_3 = 0.2191$$

$$\hat{c}_4 = 0.3029 \quad \hat{c}_5 = 0.1331 \quad \hat{c}_6 = 0.1387 \quad \hat{c}_7 = 0.0586.$$

Figure 2 compares the Bode phase plot of the $\mathcal{L}$-transform of the approximating expression (10), assuming $1+0.5\alpha \approx 1$ (curve *a*), with the phase of the expression using the usual definition for the transforms of the generalized Laguerre functions (curve *b*), and the phase $-\nu\pi/2$ of the ideal fractional integrator.

Figure 2 also shows that, near to its corner frequency $\sigma/2$, the phase of the Laguerre approximating transfer function, $g_{IN}(j\omega)$, is about equal to $-\nu\pi/2$ even if the range of frequencies around $\sigma/2$, where the phase approximates this value, is not to too wide. To enlarge this frequency range, one could increase the order N of the approximation $g_{IN}(j\omega)$.

However, more terms in the Laguerre approximations can produce transfer functions which are sensitive to the polynomial coefficient variations.

Alternatively, cascading the yet obtained $g_{IN}(j\omega)$ with a low order link achieves this goal while reducing the complexity of the compensator implementing $s^{-\nu}$. Namely, using a common lead-lag element enhances the performance of the compensator in a simpler way. The basic idea, indeed, is to choose the lag portion of the compensator to add lags on the left of the corner frequency $\sigma/2$ and use the lead portion to add phase leads on the right of $\sigma/2$. In this way, a flat, nearly constant phase of $-\nu\pi/2$ is obtained in a wide frequency range.

Figure 3 reports the Bode phase plot of: *a*) the theoretical fractional integrator; *b*) the approximation $g_{IN}(j\omega)$; *c*) the lead-lag element; *d*) the overall compensator which approximates the fractional integrator.

In figure 4, one can see the Bode magnitude plot of the fractional integrator transfer function (dashed curve *a*) and the Laguerre approximation (solid curve *b*). The dash-dotted curve *c* represents the magnitude plot of the irrational approximating function if not using the assumption to get rational functions from the transforms of the Laguerre functions.

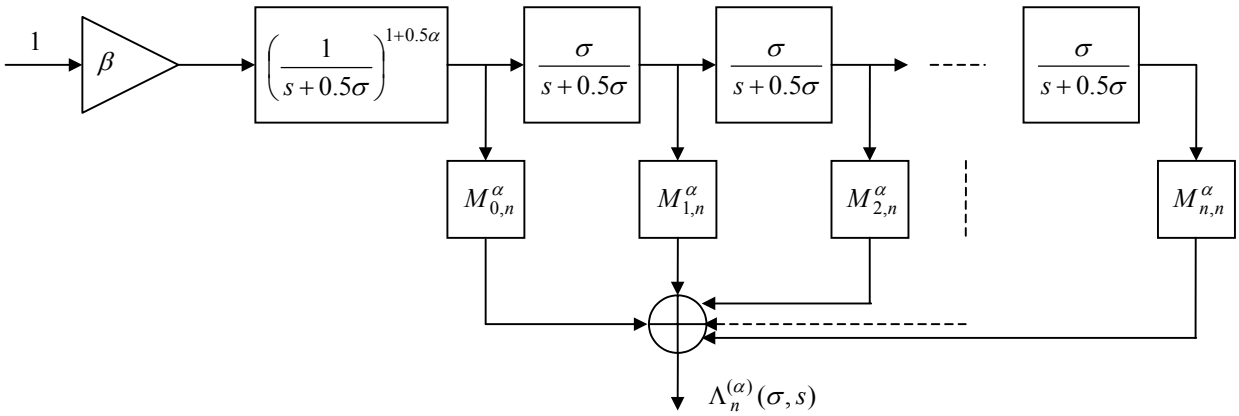


Figure 1. Network to generate the generalized Laguerre functions for even values of α.

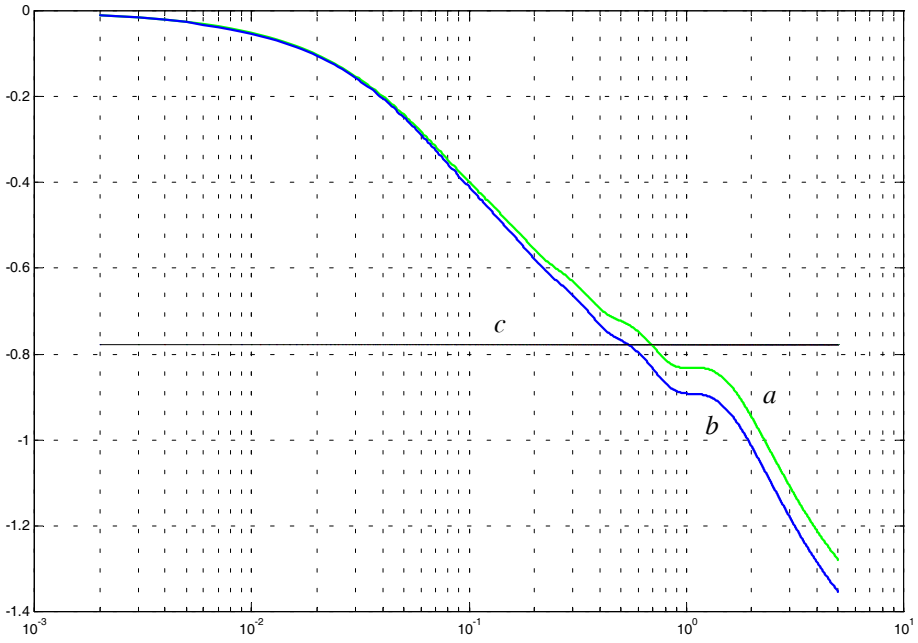


Figure 2. Bode phase plot of: *a*) the approximation $g_{IN}(j\omega)$, assuming $1+0.5\alpha \approx 1$; *b*) the transform of the generalized Laguerre function; c) the ideal fractional integrator.

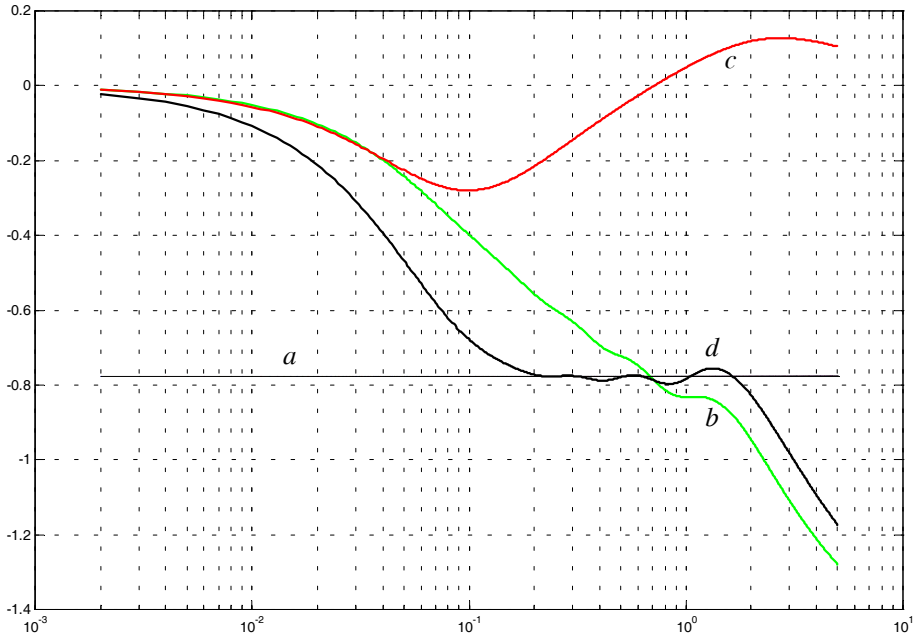


Figure 3. Bode phase plots of: *a*) the theoretical fractional integrator; *b*) the approximation $g_{IN}(j\omega)$; *c*) the lead-lag element; *d*) the overall compensator.

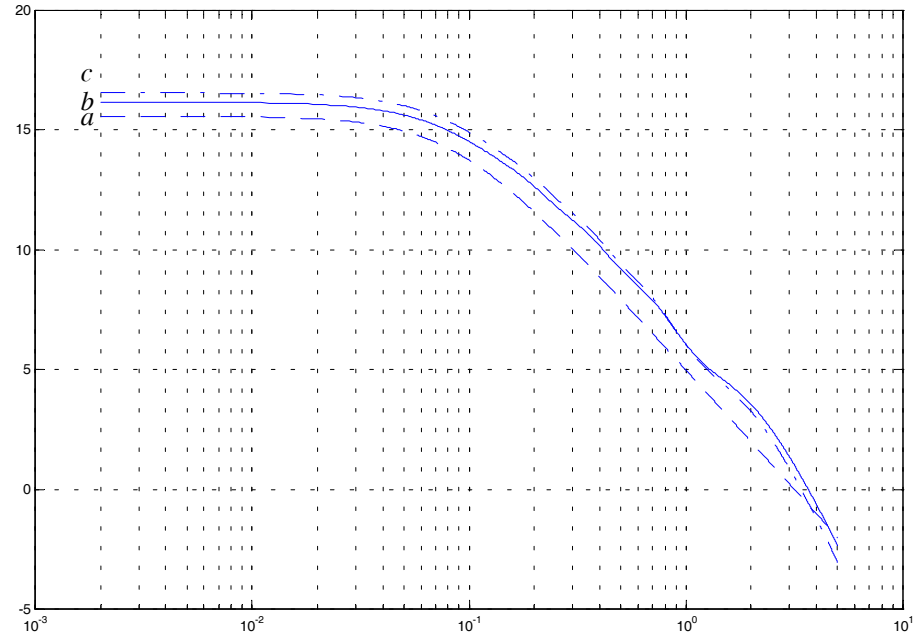


Figure 4. Bode magnitude plots of: *a*) the fractional integrator transfer function; *b*) the Laguerre rational approximation; *c*) the irrational approximating function.

4. Conclusion

This paper has suggested an approach for the realization of the irrational transfer function $g_I(s) = \Gamma(v)\,(s+a)^{-v}$, with $0<v<1$. Combining orthogonal generalized Laguerre functions and a phase shaping step, the approach leads to a simple rational approximation. The procedure is quite accurate and straightforward. An example of implementation is given.

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