Algebraic Aspects of Fuzzy Systems

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1 Fuzzy Systems and Fuzzy Relations

In performing complex tasks, very often, we recur to rules. In following rules we can recognize the general paradigm "given a condition do an action". We can code, in a given formal language, the above paradigm as a conditional statement like "if A then B", where A forms a conditional part, while B is the action part of the rule. In such a way, we can formalize knowledge based system and rule based system, in some formal language. When such rule based systems face with imprecise phenomena, it seems natural to use the formalism of fuzzy set theory. We refer to them as fuzzy rule based system.

Assuming $[0,1]$ to be the truth-values set of a given fuzzy logic we need to endow $[0,1]$ with an algebraic structure suitable to be a prototype algebraic model for such a given fuzzy logic. Here we shall limit ourselves to the case when $[0,1]$ has the MV-algebraic structure. Indeed we $[0,1]$ is endowed with the following operations:

1. $\mathbf{0} = 0$;
2. $\mathbf{1} = 1$;
3. $x \oplus y = min\{1, x + y\}$;
4. $x \odot y = max\{0, x + y - 1\}$;
5. $x^* = 1 - x$.

It results that the structure $([0,1], \odot, \oplus, *, 0, 1)$ is an MV-algebra (see section 2). It is easy to check that the set $[0,1]^X$, for an arbitrary non empty set X, is an MV-algebra too, when the operations are performed pointwise. We say that $[0,1]^X$ is the MV-algebra of all fuzzy sets over X.

So, going back to fuzzy rule based systems, when a fuzzy condition part, of a given fuzzy rule, is described by a fuzzy set $A(x)$ and the action part is described

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by another fuzzy set $B(y)$, we can think about a functional link between the fuzzy set $A(x)$ and the fuzzy set $B(y)$. The most popular model used to codify such a functional link is made by a fuzzy relation $R(x, y)$, via the equality

$$B(y) = (A \circ R)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)). \qquad (1)$$

Then we speak about a *composition operation* between a fuzzy set $A(x)$ and a fuzzy relation $R(x, y)$.

Let X, Y be sets, we recall that a (binary) *fuzzy relation* R on $X \times Y$ is a fuzzy set on $X \times Y$, i.e. $R \in F(X \times Y)$. If $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ are finite sets, then a binary classical relation R can be represented by a (binary) $\{0, 1\}$-matrix with n rows and m columns, $M_R = [r_{ij}]$ such that $(x_i, y_j) \in R$ iff $r_{ij} = 1$. Similarly a fuzzy relation $R \in F(X \times Y)$, in the case X and Y are finite sets, can be represented by a (fuzzy) binary $[0, 1]$-matrix.

Our aim is to clarify the algebraic motivations supporting the choice of (1). Although fuzzy logic provides a basis for the approximate description of different dependencies, including, of course, non linear dependency, we will see how the above formula can be understood as a model of a fuzzy function and moreover we shall show that the fuzzy systems, modelled as above, under suitable additional conditions of continuity, are linear in a specific and precise sense. Actually, to display the linearity of fuzzy rules we make use of machinery of semimodules theory in the MV-algebraic setting. The mentioned linearity evokes the simplicity informally claimed in the literature of the description of phenomena making use of fuzzy rules. In our context linearity means natural morphism between algebraic structures of same type. In fact we will show that using the semimodule structure living in the MV-algebraic setting, some dependencies, in *prima facie*, non linear, i.e., non linear in the pure MV-algebraic language, become linear in the MV-semimodule framework. An analogous phenomenon can be found in Maslov's idempotent analysis, see [].

2 Fuzzy Systems as functionals

Let $([0, 1], \oplus, \odot, ^*, 0, 1)$ the standard MV-algebra, and $R \in F(X \times Y)$ be a fuzzy relation. Then we define *domain* of R the fuzzy set $\delta_R(x) = \bigvee_{y \in Y} R(x, y)$, similarly we define the *codomain* of R the fuzzy set $\gamma_R(y) = \bigvee_{x \in X} R(x, y)$. Moreover, given two fuzzy sets $A \in f(X)$ and $B \in f(Y)$ we define the *cartesian* $\odot$- *product* of the fuzzy sets A and B as the fuzzy relation $(A \times_\circ B) \in F(X \times Y)$ defined by:

$$(A \times_\odot B)(x, y) = A(x) \odot B(y).$$

Given a fuzzy set $A \in f(X)$ and a fuzzy relation $R \in F(X \times Y)$, a relevant role in fuzzy system is played by a "partial" codomain of the cartesian $\circ$-product $A \times_\circ R$, i.e. the fuzzy set $B \in f(Y)$, such that

$$B(y) = \gamma_{A \times_\odot R}(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)) = A(x) \odot B(y).$$

Such a fuzzy subset shall be called *image* of A by R.

The above equality is often presented as an operation (composition) between a fuzzy set $A(x)$ and a fuzzy relation $R(X \times Y)$, written as:

$$B(y) = (A \odot R)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)).$$

Then, given R, the last equality can be seen as defining a function f mapping every element $A(x) \in F(X)$ to an element $B(y) \in F(Y)$ via

$$B(y) = f(A(x)) = \bigvee_{x \in X} (A(x) \circ H(x, y)).$$

In order to explain the above option for the composition between a fuzzy set $A(x)$ and a fuzzy relation $R(X \times Y)$, we will show that the above definition can be understood as a generalization of the classical concept of function. Indeed, let g be a function from the finite set $X = \{x_1, \ldots, x_n\}$ to finite set $Y = \{y_1, \ldots, y_m$ and let $G = \{(x, g(x)), x \in X\} \subset X \times Y$ be the graph of g. It happens that $(x, y) \in G$ iff $y = g(x)$. Hence the graph of g is a (boolean) binary relation, defined by:

$$I(x, y) = 1, \qquad iff \qquad y = g(x).$$

Thus, by identifying every $x_i \in X$ and every $y_j \in Y$ with its characteristic function, χ_{x_i} and χ_{y_j} respectively, we obtain:

$$\chi_{g(x_i)}(y) = \bigvee_{x \in X} \chi_{x_i}(x) \circ I(x, y)).$$

3 MV-algebras and fuzzy sets

An MV-algebra is an algebraic structure $A = (A, \oplus, ^*, 0)$ of type (2,1,0) satisfying the following axioms:

(1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(2) $x \oplus y = y \oplus x$;

(3) $x \oplus 0 = x$;

(4) $(x^*)^* = x$;

(5) $x \oplus 0^* = 0^*$;

(6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Therefore, if we define the constant 1 by $1 = 0^*$ and the operation $\odot$ by $x \odot y = (x^* \oplus y^*)^*$, then from (4), we obtain $1^* = 0$. Moreover, setting $y = 1$ in (6), it follows $x^* \oplus x = 1$. On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = (x^* \oplus y)^* \oplus y$ and $x \wedge y = (x^* \odot y)^* \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and we will refer it as the reduct lattice of the MV-algebra A. We shall write $x \leq y$ iff $x \wedge y = x$.

Proposition 1. *The reduct lattice of an MV-algebra A is resituated with respect $\odot$ operation and it results $a \to b = a^* \oplus b$, for every $a, b \in A$.*

These algebras, originating from an algebraic analysis of Łukasiewicz many-valued logic, are non-idempotent generalizations of Boolean algebras. Indeed, Boolean algebras are just the MV-algebras obeying the additional equation $x \oplus x = x$. Let $B(A) = \{x \in A \mid x \oplus x = x\}$ be the set of all idempotent elements of A. Then, $B(A)$ is a subalgebra of A, which is also a Boolean algebra. Indeed, it is the greatest Boolean subalgebra of A. Indeed, it is the greatest Boolean subalgebra of A.

Ever since their introduction by C. C. Chang, MV-algebras have been studied with respect to their relationship to other parts of mathematics as well as to their various structures. Mundici [**?**] proved there exists an equivalence functor Γ between the category of MV-algebras and the category of lattice-ordered abelian groups (abelian ℓ -groups) with strong unit. For every abelian ℓ-group with strong unit $(G, +, 0, u)$ the functor Γ equips the unit interval $[0, u]$ with the following operations:

1. $\mathbf{0} = 0$;
2. $\mathbf{1} = u$;
3. $x \oplus y = u \wedge (x + y)$;
4. $x \odot y = 0 \vee (0, x + y - 1)$;
5. $x^* = 1 - x$.

It is easy to see that the resulting structure $A = ([0, u], 0, u, *, \odot, \oplus))$ is an MV-algebra. The variety $\mathcal{MV}$ of MV-algebras coincides with the variety $HSP([0, 1])$, generated by the MV-algebra $\Gamma(R, 0, +, 1)$, where $(R, 0, +)$ is the totally ordered group of the real numbers. Thus MV-operations are defined as follows:

1. $\mathbf{0} = 0$;
2. $\mathbf{1} = 1$;
3. $x \oplus y = min\{1, x + y\}$;
4. $x \odot y = max\{0, x + y - 1\}$;
5. $x^* = 1 - x$.

Let Z be the totally ordered additive group of integers and N be the set of all the positive integers. For every $n \in N$, we shall set $S_n = \Gamma(Z, 0, +, n) = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, we shall write nx instead of $x \oplus \cdots \oplus x$ (n-times) and x^n instead of $x \odot \cdots \odot x$ (n-times). Moreover we set $0x = 0$.

Definition 2. A semiring $R = (R, +, \bullet, 0_R, 1_R)$ is an algebraic structure such that:

1. $(R, +, 0_R)$ is a commutative monoid,
2. $(R, \bullet, 1_R)$ is a monoid,
3. $\bullet$ distributes over $+$,
4. $0_R \bullet r = r \bullet 0_R = 0_R$, for every $r \in R$.

A semiring is called commutative if $(R, \bullet, 1_R)$ is commutative

Proposition 3. *Let A be an $MV-$algebra. Then the reducts $(A, \vee, \odot, 0, 1)$ and $(A, \wedge, \oplus, 1, 0)$ are commutative semirings.*

Definition 4. Let $R = (R, +, \bullet, 0_R, 1_R)$ be a semiring. A left R-semimodule is a commutative monoid $(M, +_M, 0_M)$ for which there is defined a scalar multiplication $R \times M \to M$, denoted by $(r, m) \to rm$, satisfying, for all $r, r' \in R$ and $m, m' \in M$, the followings:

1. $(r \bullet r')m = r(r'm)$,
2. $r(m +_M m') = rm +_M rm'$,
3. $(r + r')m = rm +_M r'm$,
4. $1_R m = m$,
5. $r0_M = 0_M = 0_R m$.

The definition of right R-semimodule is analogous, where the scalar multiplication is defined as a function $M \times R \to M$. An R-bisemimodule is a both right and left R-semimodule such that $(rm)r' = r(mr')$.

Example 5. *1)Let $A = (A, \oplus, \odot, ^*, 0, 1)$ and $= B(B, \oplus, \odot, ^*, 0, 1)$ be MV-algebras, $A = (A, \vee, \odot, 0, 1)$ the semiring reduct of A and $B = (B, \vee, 0)$ the sup monoid reduct of B. Moreover let h be an MV-homomorphism from A to B. Then, defining the scalar multiplication, for all $a \in A$ and $b \in B$, by:*

$$ab = h(a) \odot b,$$

we get $B = (B, \vee, 0)$ as an example of A-semimodule (MV-semimodule).

2)Let D be the free DMV-algebra over n generators [5] and and $Q' = (([0, 1] \cap Q), \vee, \cdot, 0, 1)$ the semiring, defined on the rational numbers, where " $\cdot$ " is the ordinary product. Then the reduct monoid $(D, \vee, 0)$ is a Q'-semimodule.

If X is a non-empty set and A is an MV-algebra, then the set A^X of all the functions $f : X \to A$ becomes an MV-algebra, by pointwise application of the operations defined on A.

For every $a \in A$, the constant function $x \in X \to a$ of A^X, shall be denoted by $\widetilde{a}$. In the sequel we will write $a \odot f$ instead of $\widetilde{a} \odot f$, for every $a \in A$ and $f \in A^X$.

Then we get:

Proposition 6. *Let A be an MV-algebra. Let $R = (A, \wedge, \oplus, 1, 0)$ and $R' = (A, \vee, \odot, 0, 1)$ be the reduct semirings and X a non-empty set. Then we have:*

1. *the monoid $(A^X, \vee, 0)$ is a R-bisemimodule with respect a scalar multiplication, defined by $af = a^* \odot f$, for every $a \in A$ and $f \in A^X$.*
2. *the monoid $(A^X, \vee, 0)$ is a R'-bisemimodule with respect a scalar multiplication, defined by $af = a \odot f$, for every $a \in A$ and $f \in A^X$.*

4 MV-semimodules Homomorphisms

Definition 7. Let R be a semiring and $(M, +_M, 0_M)$ and $(N+_N, 0_N)$ R-semimodules. A function H from M to N is an R-homomorphism if the following hold:

1. $H(m +_M m') = H(m) +_N H(m')$, for all $m, m' \in M$.
2. $H(rm) = rH(m)$, for all $m \in M$ and $r \in R$

The *kernel* of H is $KerH = \{m \in M : H(m) = 0_N\}$. $KerH \neq \emptyset$, since, choosing $r = 0_R$ in 2, by (5) of Definition 4, we obtain $H(0_M) = 0_N$.

Let $\mathrm{Hom}(M, N)$ be the set of all the R-homomorphisms of M in N.

$\mathrm{Hom}(M, N) \neq \emptyset$, indeed the function $0_H : m \in M \to 0_N$ is an element of $\mathrm{Hom}(M, N)$. Define on $\mathrm{Hom}(M, N)$ the operation $+_H$, by :

$$(H +_H K)(m) = H(m) +_N K(m),$$

for every $H, K \in \mathrm{Hom}(M, N)$ and $m \in M$.

Moreover, define the scalar multiplication, by:

$$(r \circ_H H)(m) = rH(m),$$

for every $H \in \mathrm{Hom}(M, N)$, $m \in M$ and $r \in R$.

With above notations we have:

Lemma 8. *If R is a commutative semiring, then $H +_H K$ and $s \circ_H H$ are elements of $Hom(M, N)$, for every $s \in R$.*

Theorem 9. *If R is a commutative semiring, then $(Hom(M, N), +_H, 0_H)$ is an R-semimodule.*

Definition 10. A nonempty subset N of a left R-semimodule M is called a *subsemimodule* if N is closed under addition and scalar multiplication.

Corollary 11. *Let A be an MV-algebra, $R = (A, \wedge, \oplus, 1, 0)$ the reduct semiring and $(A^X, \vee, 0)$, $(A^Y, \vee, 0)$ R-semimodules associated to A. Then $(Hom(A^X, A^Y), \vee_H, 0_H)$ is an R-semimodule.*

Definition 12. An element $H \in Hom(A^X, A^Y)$ is called *sup-continuous* if , for every family $\{f_i : i \in I\}$ of elements in A^X such that $\bigvee_{i\in I} f_i$ exists, $H(\bigvee_{i\in I} f_i) = \bigvee_{i\in I} H(f_i)$.

We will denote by $Hom_{sc}(A^X, A^Y)$ the set of all the sup-continuous homomorphism from A^X to A^Y. $Hom_{sc}(A^X, A^Y) \neq \emptyset$, indeed $0_H \in Hom_{sc}(A^X, A^Y)$.

Proposition 13. *$Hom_{sc}(A^X, A^Y)$ is a subsemimodule of $(Hom(A^X, A^Y, \vee_H, 0_H))$.*

5 Composition Operation as Homomorphism

We recall that an MV-algebra is complete, if the reduct lattice $(A, \vee, \wedge, 0, 1)$ is complete.

Let $A = (A, 0, 1, \oplus, \odot, *)$ be a complete MV-algebra, and X, Y two non-empty sets. Then A^X and A^Y are two complete MV-algebras endowed with

pointwise operations. We will denote by 0_X the function identically zero on X and by 0_Y the function identically zero on Y.

By introduction, we have that $(A^X, \vee, 0_X)$ and $(A^Y, \vee, 0_Y)$ as R-semimodule with respect a scalar multiplication, defined by $af = a^* \odot f$,($f \in A^X or A^Y$,) where $R = (A, \wedge, \oplus, 1, 0)$ is a reduct semiring of A.

What above allows us to interpret the composition operation as a homomorphism between the functional spaces A^X and A^Y. Indeed, let $f \in A^X$. Set $H(R)(f)(y) = \bigvee_{x \in X} f(x) \odot R(x, y) = g(y) \in A^Y$. It is easy verifying that $H(R)$ is a homomorphism. Indeed, with the above notations, we get:

Theorem 14. *$H(R)$ is a homomorphism from the R-semimodule $(A^X, \vee, 0)$ to the R-semimodule $(A^Y, \vee, 0)$.*

Now we want to study $kernH(R)$. By remark below Definition 6, $H(R)(0_X) = 0_Y$. Thus $0_X \in kernH(R) \neq \emptyset$.

We observe that the equation $\xi(x) \circ R(x, y) = 0_Y$ (with respect to unknown $\xi(x)$) is solvable, thus,by, Sanchez , $\hat{g}(x) = \bigwedge_{y \in Y} R^*(x, y)$ is the greatest solution. On the other hand, if $f(x) \leqslant g(x)$, we get $H(R(f)) \leqslant H(R(g)) = 0_Y$, which implies $H(R(f)) = 0_Y$.

What above allows us to enunciate the following

Proposition 15. *$kernH(R) = \{f \in A^X : 0_X \leqslant f \leqslant \hat{g}(x)\}$.*

Then we shown that a fuzzy composition bears a homomorphism. Next question concerns with which homomorphisms can be represented by a fuzzy composition. That is: given a homomorphism H from A^X to A^Y, which conditions on H sure us there exists a fuzzy relation $R(x, y)$ such that $H = H(R)$. When such a relation exists, we say that H is *representable by a relation*, or , shortly, *representable*. To answer last question, let us introduce some notations. For every $a \in X$, denote by f_a the element of A^X, defined as follows:

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \ , \\ 0 & \text{otherwise.} \end{cases}$$

Let $H \in Hom(A^X, A^Y)$; for every $(x, y) \in X \times Y$, set $R_H(x, y) = (H(f_x))(y)$.

With above notations, we get:

Theorem 16. *If $H \in Hom(A^X, A^Y)$ is sup-continuous, then H is representable and it results $H = H(R_H)$.*

By Proposition 5 and Corollary 10, $(A^{X \times Y}, \vee, 0)$ and $Hom(A^X, A^Y)$ are R-semimodule on the semiring $(A, \wedge, \oplus, 1, 0)$.

With above notations we get:

Theorem 17. *The function $\mathcal{H} : R \in A^{X \times Y}, \rightarrow H(R) \in Hom(A^X, A^Y)$, is an injective homomorphism from the R-semimodule $(A^{X \times Y}, \vee, 0)$ in the R-semimodule $(Hom(A^X, A^Y), \vee_H, 0_H)$*

In Theorem 17, we proved that, up an isomorphism, $\mathcal{H}(A^{X \times Y})$ is a submodule of $Hom(A^X, A^Y)$. Moreover, by Theorem 16, $\mathcal{H}(A^{X \times Y})$ is the submodule of all continuous homomorphisms of A^X to A^Y. Then we have:

Corollary 18. *The R-semimodules $(A^{X \times Y}, \vee, 0)$ and $Hom_{sc}(A^X, A^Y)$ are isomorphic.*

Corollary 19. *If X, Y are finite sets, then the function $\mathcal{H} : R \in A^{X\times Y}, \rightarrow H(R) \in Hom(A^X, A^Y)$, is an isomorphism from the R-semimodule $(A^{X\times Y}, \vee, 0_{X\times Y})$ in the R-semimodule $Hom(A^X, A^Y)$.*

6 Łukasiewicz's Transform

In the literature of fuzzy sets many extensions of the concept of *transform* are proposed and used. Such objects are called *fuzzy transform*. All the used techniques about fuzzy transforms are addressed to *discretize*, in a suitable way, a continuous function by means some primitive discrete functions. The discretization has to be a representation of the original function. It is clear that there is room for an algebraic version of fuzzy transforms. Indeed, the aim of this section is to convert a function of A^X in a n-vector. Let $A = (A, 0, 1, \oplus, \odot, *)$ be a complete MV-algebra and X two non-empty set. By above $(A^X, \vee, 0_X)$ is an R-semimodule with respect a scalar multiplication, defined by $af = a^* \odot f$,($f \in A^X or A^Y$,) where $R = (A, \wedge, \oplus, 1, 0)$ is a reduct semiring of A. Thus, if X is a finite set and $|X| = n$, we have that also $(A^n, \vee, 0_n)$ is an R-semimodule.

Now we apply the above results to the MV-algebra $A = ([0, 1], 0, 1, *, \odot, \oplus))$ and $X = [0, 1]$. Thus A^X is the set of all functions from $[0, 1]$ to $[0, 1]$, (A^n is the set of all the n-vectors valued on $[0, 1]$ and R is is a reduct semiring $([0, 1], \wedge, \oplus, 1, 0)$.

Definition 20. Let $A = \Gamma(G, +, 1)$ be an MV- algebra. A finite sequence of elements of A, $(a_0, \cdots, a_{n-1})$ is a partition of the unity 1 if $a_1 + \cdots + a_{n-1} = 1$.

Proposition 21. *Let $(A, \odot, \oplus, ^*, 0, 1)$ be an MV- algebra. Then $(a_0, ..., a_{n-1})$ is a partition of 1 if and only if $a_1 \oplus \cdots \oplus a_{n-1} = 1$ and $a_i \odot a_j = 0$, for $i, j = 0, 1, \cdots, n-1$ and $i \neq j$.*

Let us describe a particular partition of $([0, 1]^{[0,1]}, \oplus, \odot, 1_{[0,1]}, 0_{[0,1]})$.
Assume $(a, b) \in N \times Z$ and set:

if $b < 0$, then $\pi^a_b(x) = x \oplus x^*$;

if $b \geq a$, then $\pi^a_b(x) = x \odot x^*$;

if $0 \leq b \leq a - 1$, then we set:

$\pi^a_0(x) = ax$,

$\pi^a_1(x) = \bigoplus_{i=1}^{a-1} F_{0i}(x)$,

. . . . . ,

$\pi^a_b(x) = \bigoplus_{i=b}^{a-1} F_{0,1,\ldots,b-1,i}(x)$,

. . . . . ,

$\pi^a_{a-1}(x) = F_{0,1,\ldots,a-2,a-1}(x)$,

where $F_{0,1,\ldots,b-1,i}(x)$ are defined as follows:

for every integer $i > 0$, $F_{0,i}(x) = x \odot (ix)$,

for every integer $i > 1$, $F_{0,1,i}(x) = (F_{0,1}(x) \oplus \ldots \oplus F_{0,i-1}(x)) \odot F_{0,i}(x)$,

and, by induction,

for every integer i such that $i > b$,

$F_{0,1,\ldots,b,i}(x) = (F_{0,1,\ldots,b-1,b}(x) \oplus \ldots \oplus F_{0,1,\ldots,b-1,i-1}(x)) \odot F_{0,1,\ldots,b-1,i}(x)$.

For $n > 1$ set $p_k(x) = \pi_{k-1}^{n-1}(x) \wedge (\pi_k^{n-1}(x))^*$, $k = 0, \cdots n-1$.

By (P1)-(P6) and Lemma 14 of [], it follows that $(p_0(x), \cdots, p_{n-1}(x))$ is of the unity of $([0,1]^{[0,1]}, \oplus, \odot, 1_{[0,1]}, 0_{[0,1]})$.

It bears a partition of $[0,1]$ by the nodes $\{x_1, \cdots x_n\}$, where $x_k = \frac{k}{n-1}$, $k = 0, \cdots n-1$.

In analytical form we have,

$$p_0(x) = \begin{cases} -(n-1)x + 1 & \text{if } 0 \leqslant x \leqslant \frac{1}{n-1} \ , \\ 0 & \text{otherwise.} \end{cases}$$

$$p_{n-1}(x) = \begin{cases} (n-1)x - (n-2) & \text{if } \frac{n-2}{n-1} \leqslant x \leqslant 1 \ , \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 1, \cdots n-2$,

$$p_k(x) = \begin{cases} (n-1)x - (k-1) & \text{if } \frac{k-1}{n-1} \leqslant x \leqslant \frac{k}{n-1} \ , \\ -(n-1)x + k + 1) & \text{if } \frac{k}{n-1} \leqslant x \leqslant \frac{k+1}{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 22. If $x = \frac{k}{n-1}$, then $p_k(x) = 1$ and $p_h(x) = 0$, for $h \neq k$.

If $\frac{k-1}{n-1} < x < \frac{k}{n-1}$, then $p_{k-1}^*(x) = p_k(x) \neq 0, 1$ and $p_h(x) = 0$, for $h \neq k-1, k, k+1$.

If $\frac{k}{n-1} < x < \frac{k+1}{n-1}$, then $p_k^*(x) = p_{k+1}(x) \neq 0, 1$ and $p_h(x) = 0$, for $h \neq k-1, k, k+1$.

Let R be an MV-semiring and $(M, +, 0_M)$ and $(N, +, 0_N)$ R-semimodules. We call Łukasiewicz's transform of order n a homomorphism H_n from M to N such that there exists a homomorphism Λ_n from N to M having the following property:

1. $H_n \Lambda_n H_n = H_n$,
2. $\Lambda_n H_n \Lambda_n = \Lambda_n$.

If, in addition, the following condition holds:

(3) $\Lambda_n H_n(A^k)) = A^k$, for every $k = 0, 1, \cdots, n-1$,
where $\{A^k\}_{k=0,1,\cdots,n-1}$, is a given MV-partition of M,

then the Łukasiewicz's transform of order n will be called *strong*.

Define H_n by:

$$H_n : f \in [0,1]^{[0,1]} \rightarrow$$

$$\rightarrow (\bigvee_{x \in [0,1]} f(x) \odot p_0(x), \cdots, \bigvee_{x \in [0,1]} f(x \odot p_{n-1}(x))) \in [0,1]^n.$$

Setting $R(x, y_k) = p_k(x)$, for $x \in [0,1]$ and $k = 0, \cdots n-1$, we observe that H_n is a fuzzy composition; thus we have:

Theorem 23. *H_n is a homomorphism from the R-semimodule $([0,1]^{[0,1]}, \vee, 0_{[0,1]})$ to the R-semimodule $([0,1]^n, \vee, 0_n)$,*

Moreover, since H_n is a $\bigvee - \bigodot$-composition, it is a *residuated* map[]. Thus its residual map Λ_n is isotone and satisfies the following properties:

$$H_n \circ \Lambda_n \circ H_n = H_n,$$

$$\Lambda_n \circ H_n \circ \Lambda_n = \Lambda_n.$$

By [] $\Lambda_n : (v_0, \cdots, v_{n-1}) = (\bigvee_{k=0}^{n-1} v_k p_k)^*$, where the product $v_k p_k$ means the external product $v_k^* \odot p_k$.

In a certain sense, we can say, the vector $H_n(f)$ represents the components of f with respect the *base* $p_0(x), \cdots, p_{n-1}(x)$. While Λ_n tell us the way as linearly performing the elements of the base $p_0(x), \cdots, p_{n-1}(x)$ by the n-tuple of scalars $v_0, \cdots, v_{n-1}$. Indeed consider $H_n(p_k(x))$. It is equal to the vector e_k, having all the components 0, except the k-th, which is 1. Hence $\Lambda_n(e_k) = (\bigvee_{i=0, i\neq k}^{n-1} p_k^*)^*$

Generally [see[5]] the residual application $f^\sharp$ of a resituated map f satisfies the condition

$$f \circ f^\sharp \leqslant I,$$

where I is the identity. In this case we get:

Theorem 24. $H_n \circ \Lambda_n = I$

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