Strong connectivity

- Nodes $u$ and $v$ are **mutually reachable** if there is a path from $u$ to $v$ and a path from $v$ to $u$
- A directed graph is **strongly connected** if every pair of nodes are mutually reachable

**Lemma**

*Let $s$ be any node in $G$. $G$ is strongly connected $\iff$ every node is reachable from $s$ and $s$ is reachable from any node.*

**Proof.**

$\Rightarrow$ follows directly from the definition of strongly connected $G$. $\Leftarrow$ follows by constructing two paths:

- a path from $u$ to $v$ as $p = (u, \ldots, s, \ldots, v)$, and
- a path from $v$ to $u$ as $q = (v, \ldots, s, \ldots, u)$. 
Determining strong connectivity

- Select any node \( v \in V \)
- Use BFS on \( G \) from \( v \) and check if all of \( V \) is reached
- Construct \( G^r \) from \( G \) by reversing all edges
- Use BFS on \( G^r \) from \( v \) and check if all of \( V \) is reached
- If all of \( V \) is reached in both searches, \( G \) is strongly connected
- \( O(n + m) \)
- We will next see Tarjan’s algorithm which instead uses depth first search and also lists all the strongly connected components
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum +1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← ∅
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
    end
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← Ø
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
    end
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))
        if (lowlink(v) = dfn(v))
            scc ← ∅
            do
                w ← pop()
                add w to scc
            while (w ≠ v)
            process_scc(scc)
        end
    end
Tarjan's Algorithm: Initial Processing of 3

```c
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

        if (lowlink(v) = dfn(v))
            scc ← ∅
            do
                w ← pop()
                add w to scc
            while (w ≠ v)
            process_scc(scc)
    end
```

---

0, 0
1, 1
2, 2
3, 3

stack

0
1
2
3
```plaintext
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1
  for each w ∈ succ(v) do
    if (not visited(w)) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))
  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
end
```
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← ∅
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
    end

int \textit{dfnum}

\begin{verbatim}
procedure \texttt{strong\_connect}(v)  
  \textit{dfn}(v) \leftarrow \textit{dfnum}  
  \textit{lowlink}(v) \leftarrow \textit{dfnum}  
  \textit{visited}(v) \leftarrow \text{true}  
  \texttt{push}(v)  
  \textit{dfnum} \leftarrow \textit{dfnum} + 1  
  \textbf{for} each \hspace{1em} w \in \textit{succ}(v) \textbf{ do}  
    \hspace{1em} \textbf{if} (\text{not} \hspace{1em} \textit{visited}(w)) \{}  
      \hspace{2em} \texttt{strong\_connect}(w)  
      \hspace{2em} \textit{lowlink}(v) \leftarrow \min(\textit{lowlink}(v), \textit{lowlink}(w))  
    \hspace{1em} \textbf{else if} (\textit{dfn}(w) < \textit{dfn}(v) \text{ and } w \text{ is on stack})  
      \hspace{2em} \textit{lowlink}(v) \leftarrow \min(\textit{lowlink}(v), \textit{dfn}(w))  
    \}\textbf{if} (\textit{lowlink}(v) = \textit{dfn}(v))  
      \hspace{1em} \textit{scc} \leftarrow \emptyset  
      \hspace{2em} \textbf{do}  
        \hspace{3em} w \leftarrow \texttt{pop}()  
        \hspace{3em} \text{add } w \text{ to } \textit{scc}  
        \hspace{3em} \textbf{while} (w \neq v)  
          \hspace{4em} \texttt{process\_scc}(\textit{scc})  
      \hspace{2em} \textbf{end}  
    \\textbf{end}  
\end{verbatim}
Tarjan’s Algorithm: More Processing of 6

- \((6, 2) \Rightarrow 6 \text{ in same scc as } 2\).

\[
\begin{align*}
\text{int} & \quad \text{dfnum} \\
\text{procedure} & \quad \text{strong_connect}(v) \\
\text{dfn}(v) & \quad \leftarrow \text{dfnum} \\
\text{lowlink}(v) & \quad \leftarrow \text{dfnum} \\
\text{visited}(v) & \quad \leftarrow \text{true} \\
\text{push}(v) & \quad \text{dfnum} \quad \leftarrow \text{dfnum} + 1 \\
\text{for each} & \quad w \in \text{succ}(v) \text{ do} \\
\text{if (not visited}(w) & \text{) \{ \\
\text{strong_connect}(w) & \text{ \\
\text{lowlink}(v) & \quad \leftarrow \min(\text{lowlink}(v), \text{lowlink}(w)) \\
\} \text{ else if (dfn}(w) & \quad < \text{dfn}(v) \text{ and w is on stack) \\
\text{lowlink}(v) & \quad \leftarrow \min(\text{lowlink}(v), \text{dfn}(w)) \\
\text{if (lowlink}(v) & \quad = \text{dfn}(v)) \\
\text{scc} & \quad \leftarrow \emptyset \\
\text{do} & \text{ \\
\text{w} & \quad \leftarrow \text{pop}() \\
\text{add w to scc} & \text{ \\
\text{while (w} \neq & \text{v) \\
\text{process_scc}(scc)} & \text{ \\
\end{align*}
\]
(6, 3). no action.

```c
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← ∅
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
```
6 remains on the stack.

```c
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

        if (lowlink(v) = dfn(v))
            scc ← ∅
            do
                w ← pop()
                add w to scc
            while (w ≠ v)
            process_scc(scc)
        end
    end
```
Tarjan's Algorithm: More Processing of 5

- New lowlink and remains.

```c
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum +1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

        if (lowlink(v) = dfn(v))
            scc ← ∅
            do
                w ← pop()
                add w to scc
            while (w ≠ v)
            process_scc(scc)
        end
end
```
New lowlink and remains.

```c
int dfnum;

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1
    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))
    if (lowlink(v) = dfn(v))
        scc ← Ø
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
```


```plaintext
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1

  for each w ∈ succ(v) do
    if not visited(w) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))

  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
```
Tarjan’s Algorithm: Processing of 7

- Lowlink is set.

```plaintext
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← ∅
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
```

stack

jonasskeppstedt.net
Tarjan’s Algorithm: More Processing of 2

- Remove SCC from stack

```plaintext
int dfnum

procedure strong_connect(v)
    dfn(v) ← dfnum
    lowlink(v) ← dfnum
    visited(v) ← true
    push(v)
    dfnum ← dfnum + 1

    for each w ∈ succ(v) do
        if (not visited(w)) {
            strong_connect(w)
            lowlink(v) ← min(lowlink(v), lowlink(w))
        } else if (dfn(w) < dfn(v) and w is on stack)
            lowlink(v) ← min(lowlink(v), dfn(w))

    if (lowlink(v) = dfn(v))
        scc ← ∅
        do
            w ← pop()
            add w to scc
        while (w ≠ v)
        process_scc(scc)
```
No path from 2 to 8.

```
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1

  for each w ∈ succ(v) do
    if (not visited(w)) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))

  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
end
```
Tarjan's Algorithm: More Processing of 8

- 8 is its own SCC.

```plaintext
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1

  for each w ∈ succ(v) do
    if (not visited(w)) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))

  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
end
```
1 is its own SCC.

```c
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1

  for each w ∈ succ(v) do
    if (not visited(w)) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))

  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
```
0 is its own SCC.

```c
int dfnum

procedure strong_connect(v)
  dfn(v) ← dfnum
  lowlink(v) ← dfnum
  visited(v) ← true
  push(v)
  dfnum ← dfnum + 1
  for each w ∈ succ(v) do
    if (not visited(w)) {
      strong_connect(w)
      lowlink(v) ← min(lowlink(v), lowlink(w))
    } else if (dfn(w) < dfn(v) and w is on stack)
      lowlink(v) ← min(lowlink(v), dfn(w))
  if (lowlink(v) = dfn(v))
    scc ← ∅
    do
      w ← pop()
      add w to scc
    while (w ≠ v)
    process_scc(scc)
```
Consider the edge \((v, w)\).

When \(w\) is not yet visited we must visit it by calling \texttt{strong_connect}(w).

If \(w\) has been visited, we have two main cases:

1. \(w\) is not on the stack, because it has already found its SCC.
2. \(w\) is on the stack, because it’s waiting for being popped.
   - If \(dfn(w) < dfn(v)\) then \(v\) must set its lowlink so it does not think it is its own SCC.
   - If \(dfn(w) \geq dfn(v)\) then no more information for \(v\) is available. There is another path from \(v\) to \(w\) due to which they will belong to the same SCC.
It is not trivial to define precisely what makes an algorithm greedy.

The main idea is to use a simple rule to make decisions without taking "all" information into account.

The challenge is to find a simple rule which solves a problem optimally!

Two approaches to prove that a greedy algorithm is optimal:

1. The greedy algorithm "stays ahead" — by proving it is always at least as good as an optimal algorithm, we know the greedy also is optimal

2. Exchange argument (utbytesargument) — transform the output of an optimal algorithm (without changing its quality) to the output of the greedy algorithm
Interval scheduling / Intervallschemaläggning

- One resource
- A set $R$ of requests, $r_i$, with a start time $s(i)$ and a finish time $f(i)$
- A set of requests is **compatible** if they do not overlap in time
- The **interval scheduling problem** is to find the largest subset $S \subseteq R$ such that $S$ is compatible
- All requests have equal value and it is the size of $S$ we want to maximize
- A compatible set of maximum size is called an optimal schedule
An example set $R$
A greedy algorithm for interval scheduling

procedure schedule\( (R) \)
\( S \leftarrow \emptyset \) /\(^*\) S is a sequence /\(^*\)
while \( R \neq \text{null} \)
    \( r \leftarrow \text{select a request from } R \)
    remove \( r \) from \( R \)
    add \( r \) to the end of \( S \)
    remove all request in \( R \) which overlap with \( r \)
return \( S \)

- Our problem is to figure out a clever select function
- Any suggestions?
Ideas for the select function

- Take the request with shortest interval
- Take the request which starts first
- Take the request with fewest conflicts

None of these lead to an optimal solution
A better select function

- Take the request which finishes earliest
- Is this optimal?
- Select first request:
Select request which finishes first

- Remove incompatible requests:
Select request which finishes first

Select next request:
Select request which finishes first

- Remove incompatible request:
Select request which finishes first

Select next request:
Select request which finishes first

- Remove incompatible requests:
Select request which finishes first

End result:
Proving optimality of a greedy algorithm

- What should we prove?
- Can there exist several optimal schedules?
- Either show our algorithm is as good as an optimal (stays ahead), or output from an optimal algorithm can be transformed to the output of our algorithm.
- For our problem, assume there is an optimal schedule represented as a sequence \( T \) sorted in order of increasing finish time.
- We should not try to prove \( S = T \).
- Instead we should prove \( |S| = |T| \).
- We will use the first proof technique: "our algorithm stays ahead of an optimal solution”
- That is, \( |S| \geq |T| \)
Two solutions

- Our: \( S = (r_1, r_2, \ldots, r_n) \)
- Optimal: \( T = (t_1, t_2, \ldots, t_m) \)
- We want to show \( n = m \)
- \( S \) and \( T \) are both sorted by increasing finish time
- It is clear that \( f(r_1) \leq f(t_1) \) since we select the request with earliest finish time
- There are at least \( n \) requests in \( T \) so we can aim at proving:
  - \( f(r_k) \leq f(t_k) \quad 1 \leq k \leq n \)
Comparing the solutions

Lemma

\[ f(r_k) \leq f(t_k) \quad 1 \leq k \leq n \]

Proof.

- Proof by induction. \( f(r_1) \leq f(t_1) \) is clear.
- For \( k > 1 \), assume \( f(r_{k-1}) \leq f(t_{k-1}) \).
- Since \( T \) is compatible, \( f(t_{k-1}) \leq s(t_k) \).
- \( f(r_{k-1}) \leq s(t_k) \)
- Our algorithm can select \( t_k \) as its \( r_k \)
- Our algorithm selects as \( r_k \) the request with earliest finish time, i.e. \( f(r_k) \leq f(t_k) \)
It remains to prove that $|S| = |T|$. 

Recall $n = |S|$ and $m = |T|$. 

**Theorem**

$|S| = |T|$. 

**Proof.**

- Assume in contradiction that $m > n$.
- We know $f(r_n) \leq f(t_n)$
- Since $m > n$, $T$ contains a request $t_{n+1}$
- We must have $s(t_{n+1}) \geq f(t_n) \geq f(r_n)$
- But this request should have been scheduled by our algorithm which contradicts the assumption that $|S| = n$ so $m = n$ and optimality has been proved
Our greedy algorithm can be implemented in time $O(n \log n)$

First all requests are sorted in order of increasing finish time

Then a linear pass finds the schedule
Again one resource

Consider now requests with a soft deadline $d(r)$ and a time length $t(t)$

It is not a disaster to fail a soft deadline compared with a hard deadline

$s(r)$ and $f(r)$ are start and finish times and in this problem they are output and not input

The delay of one request is $\max(0, f(r) - d(r))$

Our problem is to schedule requests so that the maximum delay of any request is minimized

What is a simple rule to do that optimally?
Idle time

- Request are renamed so that \( d(r_1) \leq d(r_2) \ldots \leq d(r_n) \)
- Our algorithm simply is to schedule the requests in this order, or in other words, the earliest deadline first
- It is nice to know we can use this in practical situations in life as well!
- In addition, we schedule requests so that \( s(r_{i+1}) = f(r_i) \), i.e. without a gap between \( r_i \) and \( r_{i+1} \)
- Thus there is no idle time between any two requests
- Consider any optimal schedule \( T \). Can it have idle time?
- Yes, e.g. if \( t(r_1) = 1, d(r_1) = 2, t(r_2) = 3, d(r_2) = 10 \)
- With \( s(r_1) = 1, f(r_1) = 2, s(r_2) = 7, d(r_2) = 10 \), the maximum delay is zero
- But there obviously exists a different optimal schedule without any gap — just start the requests in the same order but as early as possible
Inversions

- Let $d(r_i) < d(r_j)$
- If $r_j$ scheduled before $r_i$, it is called an inversion
- Our algorithm creates no inversions
- But if $d(r_i) = d(r_{i+1})$ then it can schedule $r_{i+1}$ before $r_i$
Lemma

All schedules with no idle time and no inversions have the same maximum delay

Proof.

- Consider two different schedules $S$ and $T$ without idle time and no inversions
- Then the only difference between $S$ and $T$ is the order in which requests with identical deadlines are scheduled
- In a sequence with such requests, the last has the maximal delay or no delay.
- The maximal delay is the same in both schedules.
An example

- Let all requests take one time unit
- Let all requests have deadline \( d(r_i) = 3 \)
- \( S = (r_1, r_2, r_3, r_4, r_5) \)
- \( T = (r_4, r_5, r_1, r_2, r_3) \)
- In \( S \) \( r_1, r_2, r_3 \) have no delay, \( r_4 \) is delayed 1 and \( r_5 \) is delayed 2
- In \( T \) \( r_4, r_5, r_1 \) have no delay, \( r_2 \) is delayed 1 and \( r_3 \) is delayed 2
- Same maximum delay but for different requests
We will prove that an optimal schedule $T$ can be transformed to $S$.

**Lemma**

Assume an optimal schedule $Q$ has an inversion of $r_i$ and $r_j$ i.e. $d_i < d_j$ and $r_j$ is scheduled before $r_i$. Then $Q$ has a pair $r_a$ and $r_b$ of inverted requests scheduled immediately after each other.

In $(7, 8, 9, 1, 2, 3, 4, 5, 6)$ with $i = 1, j = 7$ we have $a = 1, b = 9$.

**Proof.**

- We have $(r_j, ..., r_i)$ with $k$ requests scheduled between $r_j$ and $r_i$, $k \geq 0$.
- In this sequence of $k + 2$ requests, since $d_i < d_j$, there must be a first pair $r_a$ and $r_b$ of inverted requests with $r_b$ scheduled immediately before $r_a$ such that $d_a < d_b$. 
Reducing the number of inversions

- We will in steps prove that there exists an optimal schedule $T$ without idle time and inversions.
- We have already seen there exists an optimal schedule without idle time.
- The exchange argument is to change an optimal schedule with inversions to another optimal schedule without inversions.
- We must show that the maximum delay does not increase, of course.
- From the previous lemma, we know we can remove one inversion, namely of $r_a$ and $r_b$
- Recall $d_a < d_b$, $r_b$ was scheduled immediately before $r_a$ and with the exchange they are instead scheduled with $r_a$ immediately before $r_b$.
- Can the maximum delay increase?
Lemma

Let \( r_a \) and \( r_b \) be a pair of inverted requests scheduled consecutively in \( Q \). Swapping \( r_a \) and \( r_b \) into \( R \), the maximum delay does not increase.

Proof.

- We have \( Q = (\ldots, r_b, r_a, \ldots) \) and \( R = (\ldots, r_a, r_b, \ldots) \).
- The delay of \( r_a \) cannot have increased in \( R \).
- Let \( f^X(r) \) be the finish time in schedule \( X \) for request \( r \).
- Let \( t = f^Q(r_a) = f^R(r_b) \) (i.e. finish time of last of them).
- The delay of \( r_a \) in \( Q \) is \( f^Q(r_a) - d(r_a) = t - d(r_a) \).
- The delay of \( r_b \) in \( R \) is \( f^R(r_b) - d(r_b) = t - d(r_b) \).
- Can \( r_b \) now be more late than \( r_a \) was? I.e. can \( t - d(r_b) > t - d(r_a) \)?
- \( t - d(r_b) > t - d(r_a) \iff t + d(r_a) > t + d(r_b) \).
- No, since by assumption \( d(r_a) < d(r_b) \).
Theorem

*Our algorithm produces a schedule with minimum maximum delay.*

Proof.

- We have just shown that there exists an optimal schedule $T$ with no idle time and no inversions.
- Since all schedules with no idle time and no inversions have the same maximum delay, our algorithm is optimal.
An exchange argument

What did we do?
We find an algorithm which seems to be optimal.
We characterize optimal solutions.
We exchange optimal solutions to produce an output which is identical to ours.
Therefore our algorithm is optimal.
Note that the schedules may be different but the minimum maximum delay is the same.