Lecture 6: A graph recurrence. Convolutions

Given a fan:

How many spanning trees \( f_n \) are there?
Can choose \( n \) of \( 2n - 1 \) edges in \( \binom{2n-1}{n} \) ways, but not all gives a spanning tree.

**Step 1:**
\[
f_n = f_{n-1} + f_{n-1} + f_{n-2} + f_{n-3} + \cdots + f_1 + 1, \quad n \geq 1
\]
or
\[
f_n = f_{n-1} + \sum_{k<n} f_k + [n > 0], \quad \text{for all } n
\]

**Example:** \( n = 4 \) in figure on page 349.

**Step 2:**
\[
F(z) = \sum_n f_{n-1} z^n + \sum_{k,n} f_k z^n [n > k] + \sum_{n} [n > 0] z^n
\]
\[
= z F(z) + \sum_k f_k z^k \sum_n [n > k] z^{n-k} + \frac{z}{1-z}
\]
\[
= z F(z) + F(z) \sum_{m>0} z^m + \frac{z}{1-z}
\]
\[
= z F(z) + F(z) \frac{z}{1-z} + \frac{z}{1-z}
\]

**Step 3:**
\[
F(z) = \frac{z}{1 - 3z + z^2}
\]

**Step 4:** Given \( G(z) \rightarrow \langle g_n \rangle \) produce \( \langle g_0, 0, g_2, 0, g_4, 0, \ldots \rangle \) using
\[
G(z) + G(-z) = \sum_n g_n (1 + (-1)^n) z^n = 2 \sum_n g_n z^n [n \text{ even}]
\]
that is
\[
\frac{G(z) + G(-z)}{2} = \sum_n g_{2n} z^{2n}
\]
Especially for Fibonacci numbers
\[
\sum_n F_{2n} z^{2n} = \frac{1}{2} \left( \frac{z}{1-z-z^2} + \frac{-z}{1+z-z^2} \right) = \frac{z^2}{1 - 3z^2 + z^4}
\]
Hence
\[
\sum_n F_{2n} z^n = \frac{z}{1 - 3z + z^2} \rightarrow \langle f_n \rangle
\]
Alternative derivation

\[ R(z) = \frac{P(z)}{Q(z)} = \frac{z}{1 - 3z + z^2} \]

\[ z^2 - 3z + 1 = 0 \Rightarrow z = \frac{3 \pm \sqrt{5}}{2} \]

\[ 1/\rho_1 = \frac{3 + \sqrt{5}}{2}, \quad \rho_1 = \frac{3 - \sqrt{5}}{2} \]

\[ 1/\rho_2 = \frac{3 - \sqrt{5}}{2}, \quad \rho_2 = \frac{3 + \sqrt{5}}{2} \]

\[ [z^n] R(z) = a_1\rho_1^n + a_2\rho_2^n \]

where

\[ a_k = -\frac{\rho_k P(1/\rho_k)}{Q'(1/\rho_k)}, \quad \text{with } Q' = 2z - 3 \]

\[ a_1 = -1/\sqrt{5}, \quad a_2 = 1/\sqrt{5}. \]

\[ [z^n] R(z) = \frac{1}{\sqrt{5}}\left(\left(\frac{3 + \sqrt{5}}{2}\right)^n - \left(\frac{3 - \sqrt{5}}{2}\right)^n\right) = \frac{1}{\sqrt{5}}\left((\Phi + 1)^n - (\tilde{\Phi} + 1)^n\right) \]

which generates \( \langle 0, 1, 3, 8, \ldots \rangle \)

Check: \( n = 0 \) gives \( \frac{1}{\sqrt{5}}(1 - 1) = 0 \) and \( n = 1 \) gives \( \frac{1}{\sqrt{5}}(\Phi - \tilde{\Phi}) = 1. \)

**An identity**

Show

\[ \frac{1}{(1 - z)^{m+1}} \ln \frac{1}{1 - z} = \sum_n (H_{m+n} - H_m) \binom{m+n}{n} z^n \]

The binomial theorem gives

\[ \frac{1}{(1 - z)^{x+1}} = \sum_n \binom{-x-1}{n}(-1)^n z^n = \sum_n \binom{x+n}{n} z^n \]

Take derivate wrt \( x \)

\[ \frac{d}{dx}(e^{(x+1)\ln(1/(1-z))}) = (\ln \frac{1}{1 - z}) \frac{1}{(1 - z)^{x+1}} \]

\[ \frac{d}{dx}\left(\binom{x+n}{n}\right) = \frac{1}{x+n} + \cdots + \frac{1}{x+1}\binom{x+n}{n} = (H_{x+n} - H_x)\binom{x+n}{n} \]

\( x \leftarrow m \) gives the identity above. For \( m = 0 \) we get

\[ \frac{1}{1 - z} \ln \frac{1}{1 - z} = \sum_n H_n z^n \]

Hence, \( \frac{1}{1 - z} \ln \frac{1}{1 - z} \) is GF for \( \langle H_n \rangle \).

**Convolutions**

The convolution of \( \langle f_n \rangle \) and \( \langle g_n \rangle \): \( \langle \sum_k f_k g_{n-k} \rangle \), corresponds to multiplication of GF.
A Fibonacci convolution

Evaluate \( \sum_{k=0}^{n} F_k F_{n-k} \), i.e. \([z^n] F(z)^2\). Given that

\[
\frac{1}{(1 - \Phi z)(1 - \hat{\Phi} z)} = \frac{1}{1 - (\Phi + \hat{\Phi}) z + (\Phi \hat{\Phi}) z^2} = \frac{1}{1 - z - z^2} = \frac{F(z)}{z}, \quad \text{left shift with } f_0 = 0
\]

and

\[
\Phi + \hat{\Phi} = \frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} = 1, \quad \Phi \hat{\Phi} = \frac{(1 + \sqrt{5})(1 - \sqrt{5})}{4} = -1
\]

then

\[
F(z)^2 = \left( \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \Phi z} - \frac{1}{1 - \hat{\Phi} z} \right) \right)^2
\]

\[
= \frac{1}{5} \left( \frac{1}{(1 - \Phi z)^2} - \frac{2}{(1 - \Phi z)(1 - \hat{\Phi} z)} + \frac{1}{(1 - \hat{\Phi} z)^2} \right)
\]

\[
= \frac{1}{5} \sum_{n \geq 0} (n + 1) \Phi^n z^n - \frac{2}{5} \sum_{n \geq 0} F_{n+1} z^n + \frac{1}{5} \sum_{n \geq 0} (n + 1) \hat{\Phi}^n z^n
\]

since

\[
\frac{1}{(1 - z)^2} = \sum (n + 1) z^n
\]

Together with

\[
\Phi^n + \hat{\Phi}^n = [z^n] \left( \frac{1}{1 - \Phi z} + \frac{1}{1 - \hat{\Phi} z} \right)
\]

\[
= [z^n] \frac{2 - (\Phi + \hat{\Phi}) z}{(1 - \Phi z)(1 - \hat{\Phi} z)} = [z^n] \frac{2 - z}{1 - z - z^2} = 2F_{n+1} - F_n
\]

this gives

\[
F(z)^2 = \frac{1}{5} \sum_{n \geq 0} (n + 1)(2F_{n+1} - F_n) z^n - \frac{2}{5} \sum_{n \geq 0} F_{n+1} z^n = \frac{1}{5} \sum (2nF_{n+1} - (n + 1)F_n) z^n
\]

so

\[
\sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1} - (n + 1)F_n}{5}
\]

**A harmonic convolution**

Analysis of "sample sort" results in

\[
T_{m,n} = \sum_{k=0}^{n-1} \binom{k}{m} \frac{1}{n - k} \quad \text{integer } m, n \geq 0 \text{ (note that } 1/(n - k) \text{ is undefined for } k = n)\]

Insight

\[
T_{m,n} = [z^n] \left( \binom{0}{m}, \binom{1}{m}, \ldots, 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \right)
\]

where

\[
\sum_{n \geq 0} \binom{n}{m} z^n = \frac{z^m}{(1 - z)^{m+1}} \quad \text{and} \quad \sum_{n>0} \frac{z^n}{n} = \ln \frac{1}{1 - z}
\]
so

\[
T_{m,n} = [z^n] \frac{z^m}{(1-z)^{m+1}} \ln \frac{1}{1-z}
= [z^{n-m}] \frac{1}{(1-z)^{m+1}} \ln \frac{1}{1-z}
= (H_n - H_m) \binom{n}{n-m}, \text{ by identity on page 2 of this lecture}
\]

**Convolution of convolutions**

The convolution of \(\langle f_n \rangle\) and \(\langle g_n \rangle\) convoluted with \(\langle h_n \rangle\) is

\[
[z^n] F(z)G(z)H(z) = \sum_{j+k+l=n} f_j g_k h_l
\]

Similarly, the \(m\)-fold convolution of \(\langle g_n \rangle\) with itself is

\[
[z^n] G(z)^m = \sum_{k_1+\cdots+k_m=n} g_{k_1} \cdots g_{k_m}
\]

**Apply** on the fan graph problem: If there is no edge between nodes \(j\) and \(j+1\) then they belong to different blocks. Compute the number of ways to connect the 0-node to the other blocks.

**Example** of block partition for \(n = 10\) in figure on page 356.

For \(n = 4\) we get \(f_4 = 4 + 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1 \cdot 1 = 21 = F_8\)

Number of blocks

\[
f_n = \sum_{m>0} \sum_{k_1+\cdots+k_m=n} k_1 \cdots k_m, \text{ with } k_1, \ldots, k_m > 0
\]

which is the sum of \(m\)-fold convolutions of \(\langle 0, 1, 2, 3, \ldots \rangle\) for \(m = 1, 2, 3, \ldots\), so

\[
F(z) = G(z) + G(z)^2 + G(z)^3 + \cdots = \frac{G(z)}{1-G(z)}, \text{ with } G(z) = \frac{z}{(1-z)^2}
\]

since \(\sum_{n\geq0}(n+1)z^n = \frac{1}{(1-z)^2}\) generates \(\langle 1, 2, 3, \ldots \rangle\), so

\[
F(z) = \frac{z}{(1-z)^2 - z} = \frac{z}{1-3z + z^2}
\]

which is the same as before for the fan.

**A convoluted recurrence**

If \(x_0, x_1, \ldots, x_n\) are to be multiplied together, in how many ways \(C_n\) can we insert parentheses?

For small \(n\): \(C_0 = C_1 = 1, C_2 = 2, C_3 = 5\).

**Step 1:**

\[
C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0, \: n > 0
= \sum_k C_kC_{n-1-k} + [n = 0]
\]
such that $x$

If we plot the partial sums $s_n$ we see that there are $C_m$ in the cycle from which all partial sums will be positive, because every other point on the curve has an average slope $1/m$.

Step 4: What is $[z^n]C(z)$? The binomial theorem gives

$$\sqrt{1 - 4z} = \sum_{k \geq 0} \binom{1/2}{k} (-4z)^k = 1 + \sum_{k \geq 1} \frac{1}{2k} \binom{-1/2}{k-1} (-4z)^k$$

and since $\binom{-1/2}{n} = (-1)^n / 4^n \binom{2n}{n}$, by "going halves" in Lecture 3 we have

$$\frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{k \geq 1} \frac{(-1/2)}{k} (-4z)^{k-1} = \sum_{n \geq 0} \binom{-1/2}{n} \frac{(-4z)^n}{n+1} = \sum_{n \geq 0} \frac{2n}{n+1} z^n = \sum_{n \geq 0} C_n z^n$$

Catalan numbers appear in many contexts. For instance, as the number of binary trees with $n$ nodes.

Example: How many sequences $\langle a_1, \ldots, a_{2n} \rangle$, $a_i = \pm 1$ have the property:

$$a_1 + a_2 + \cdots + a_{2n} = 0,$$

with all partial sums $a_1 \geq 0$, $a_1 + a_2 \geq 0$, $a_1 + a_2 + a_3 \geq 0$, $\ldots$

There are $C_n$ such mountain ranges, which can be related to parentheses: $+1$ corresponds to "($" and $-1$ corresponds to ")".

Variation: How many sequences $\langle a_0, \ldots, a_{2n} \rangle$, $a_i = \pm 1$ have the property that $\sum_{i=0}^{2n} a_i = 1$ and $a_0 > 0$, $a_0 + a_1 > 0$ $\ldots$? The answer again is $C_n$, derived with the help of

Raney’s lemma: If $\langle x_1, x_2, \ldots, x_m \rangle$ is any integer sequence which sums to $+1$, then exactly one of the cyclic shifts:

$$\langle x_1, x_2, \ldots, x_m \rangle, \quad \langle x_2, \ldots, x_m, x_1 \rangle, \quad \ldots, \quad \langle x_m, x_1, \ldots, x_{m-1} \rangle$$

has all of its partial sums positive.

The lemma can be proved by a geometric argument. Extend the sequence periodically to an infinite sequence

$$\langle x_1, x_2, \ldots, x_m, x_1, x_2, \ldots, x_m, x_1, x_2, \ldots \rangle$$

such that $x_{m+k} = x_k$, for all $k > 0$.

If we plot the partial sums $s_n = x_1 + \cdots + x_n$ as functions of $n$, these have an average slope $1/m$ because $s_{m+n} = s_n + 1$. The entire graph $s_n$ lies between two lines of slope $1/m$. These bounding lines touch the graph just once in each cycle of $m$ points. The unique lower intersection point is the only place in the cycle from which all partial sums will be positive, because every other point on the curve has an intersection point within $m$ units to its right.
Using Raney’s lemma we can enumerate the sequences \( (a_0, \ldots, a_{2n}) \), \( a_i = \pm 1 \) whose partial sums are positive and whose total sum is 1. There are \( \binom{2n+1}{n} \) sequences with \( n \) occurrences of \(-1\) and \( n + 1 \) occurrences of \(+1\). Raney’s lemma says that \( 1/(2n+1) \) of these all have positive partial sums, that is

\[
\binom{2n+1}{n} \frac{1}{2n+1} = \binom{2n+1}{(2n+1) - n} \frac{1}{2n+1} = \binom{2n}{n} \frac{1}{n+1} = C_n
\]