Lecture 12: Euler’s summation formula

\[ \sum_{k=a}^{b-1} f(k) = \int_a^b f(x) \, dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x) |^b_a + R_m \]

where

\[ R_m = (-1)^{m+1} \int_a^b \frac{B_m(x)}{m!} f^{(m)}(x) \, dx, \quad \text{integer } a \leq b, \ m \geq 1 \]

\( B_k \) are Bernoulli numbers, \( B_m(x) \) are Bernoulli polynomials and \( \{x\} = x - \lfloor x \rfloor \)

For instance: \( f(x) = x^{m-1} \) gives \( f^{(m)}(x) = 0 \), so \( R_m = 0 \), and

\[ \sum_{k=a}^{b-1} k^{m-1} = \frac{x^m}{m} |^b_a + \sum_{k=1}^{m} \frac{B_k}{k!} (m - 1)^{k-1} x^{m-k} |^b_a \]
\[ = \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} B_k (b^{m-k} - a^{m-k}) \]

Example:

\[ \sum_{k=0}^{n-1} k^2 = \frac{1}{3} \binom{3}{0} B_0 n^3 + \binom{3}{1} B_1 n^2 + \binom{3}{2} B_2 n \]
\[ = n^3 - \frac{n^2}{2} + \frac{n}{6} \]

Michael Spivey’s article “The Euler-Maclaurin Formula and Sums of Powers” is handed out.

Claim:

\[ 1^m + 2^m + \cdots + (m-1)^m < m^m \], for \( m \geq 1 \)

Proof using Euler’s formula with \( R_2 \) as error term

\[ \sum_{j=1}^{m-1} j^m = \int_0^m x^m \, dx - \frac{1}{2} m^m + \frac{1}{12} m \cdot m^{m-1} - \int_0^m \frac{B_2(x)}{2!} m(m-1)x^{m-2} \, dx \]
\[ = \frac{m^{m+1}}{m+1} - \frac{5}{12} m^m - \frac{1}{2} \int_0^m B_2(x) m(m-1)x^{m-2} \, dx \]

Error term has \( B_2(x) = (x-1/2)^2 - 1/12 \). Hence in \([0,1]\) its minimum value is \(-1/12\), for \( x = 1/2\), and its maximum is \(1/6\), at the two endpoints. Since \( B_2 \) is periodic these are also global extreme values, which gives \( -B_2(\{x\})/2 \leq (-1/2)(-1/12) \leq 1/24 \). This bounds the error term to

\[ -\frac{1}{2} \int_0^m B_2(\{x\}) m(m-1)x^{m-2} \, dx \leq \frac{1}{24} \int_0^m m(m-1)x^{m-2} \, dx = \frac{m}{24} m^{m-1} = \frac{m^m}{24} \]

Hence

\[ \sum_{j=1}^{m-1} j^m \leq \frac{m^{m+1}}{m+1} - \frac{5}{12} m^m + \frac{m^m}{24} < m^m - \frac{3}{8} m^m = \frac{5}{8} m^m \]

Next, determine the limiting expression for

\[ \left( \frac{1}{m} \right)^m + \left( \frac{2}{m} \right)^m + \cdots + \left( \frac{m-1}{m} \right)^m \]
We know it is \(< \frac{5}{8}\). What is the exact value?

For fixed \(m\) and \(f(x) = x^m\), Euler’s formula gives

\[
\sum_{j=1}^{m-1} \left(\frac{j}{m}\right)^m = \frac{1}{m^m} \sum_{j=0}^{m-1} j^m
\]

\[
= \frac{1}{m^m} \int_0^m x^m \, dx + \frac{1}{m^m} \sum_{k=1}^{\infty} B_k \frac{f^{(k-1)}(m) - f^{(k-1)}(0)}{k!}
\]

\[
= \frac{m}{m+1} + \frac{1}{m^m} \sum_{k=1}^{\infty} B_k \frac{f^{(k-1)}(m) - f^{(k-1)}(0)}{k!}
\]

Since \(f^{(k-1)}(m) - f^{(k-1)}(0)\) is nonzero only for \(k \leq m\), this gives

\[
\sum_{j=1}^{m-1} \left(\frac{j}{m}\right)^m = \frac{m}{m+1} + \frac{1}{m^m} \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(m)
\]

\[
= \frac{m}{m+1} + \frac{1}{m^m} \sum_{k=1}^{m} \left[\frac{B_k}{k!} m^{m-k+1} (m(m-1) \cdots (m-k+2))\right]
\]

\[
= \frac{m}{m+1} + \sum_{k=1}^{m} \left[\frac{B_k}{k!} m^{1-k} (m(m-1) \cdots (m-k+2))\right]
\]

Since \(m(m-1) \cdots (m-k+2) = m^{k-1} + O(m^{k-2})\), we have

\[
\sum_{j=1}^{m-1} \left(\frac{j}{m}\right)^m = \frac{m}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k!} [1 + O(1/m)]
\]

Taking the limit

\[
\lim_{m \to \infty} \sum_{j=1}^{m-1} \left(\frac{j}{m}\right)^m = 1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} + \lim_{m \to \infty} \left\{O(1/m) \sum_{k=1}^{m} \frac{B_k}{k!}\right\}
\]

We know from earlier in the course that

\[
\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}
\]

valid for \(|x| < 2\pi\). Therefore, \(\sum_{k=1}^{m} B_k / k!\) is bounded by a constant, and

\[
\lim_{m \to \infty} \left\{O(1/m) \sum_{k=1}^{m} \frac{B_k}{k!}\right\} = 0
\]

Since \(B_0 = 1\), we have

\[
\lim_{m \to \infty} \sum_{j=1}^{m-1} \left(\frac{j}{m}\right)^m = \sum_{k=0}^{\infty} \frac{B_k}{k!}
\]

and hence

\[
\lim_{m \to \infty} \left[\frac{1}{m} + \frac{2}{m} + \cdots + \frac{m-1}{m}\right] = \frac{1}{e - 1} \approx 0.582
\]

attained from below.