Type Systems
02 – Basic Concepts

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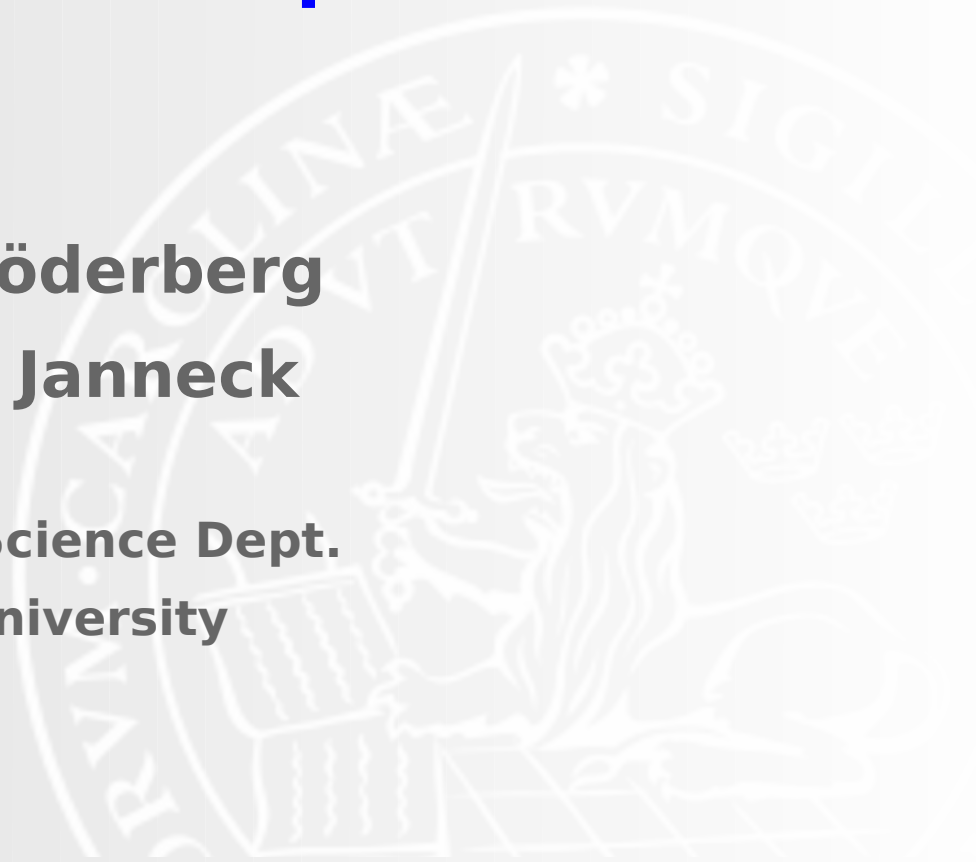
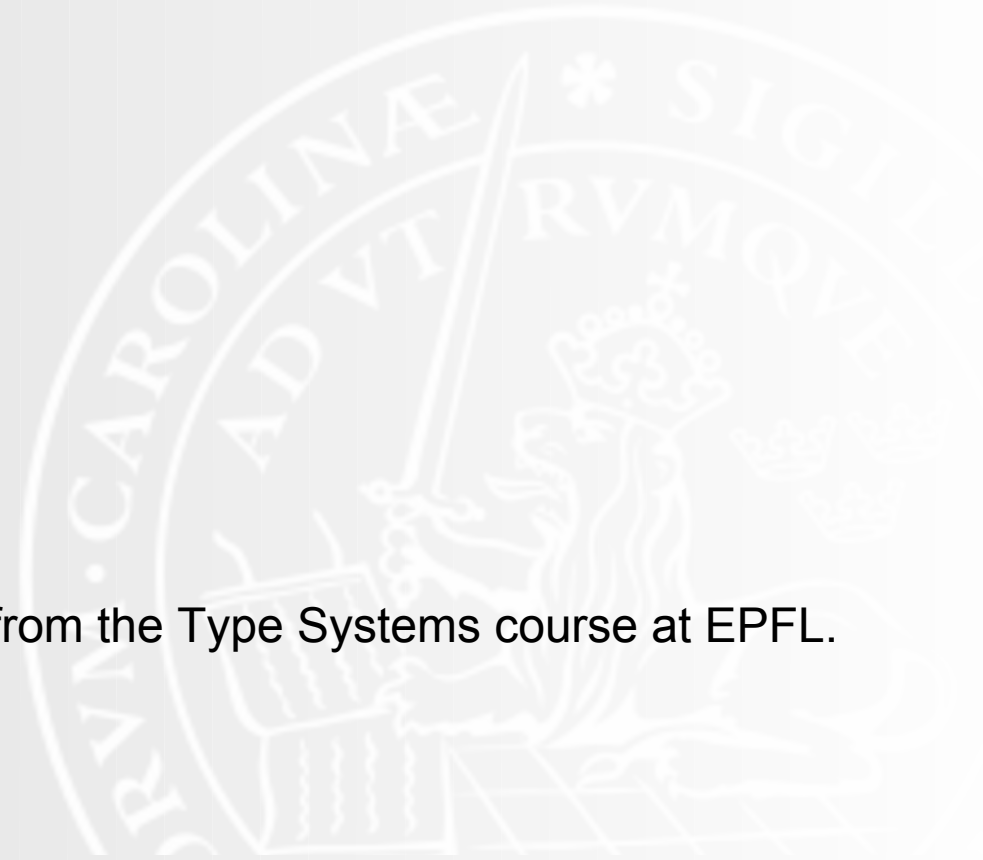


Slides begged, borrowed, and copied from the Type Systems course at EPFL. Courtesy of Martin Odersky.



Simple Arithmetic Expressions

Simple Arithmetic Expressions

Here is a BNF grammar for a very simple language of arithmetic expressions:

`t` ::=		*terms*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
	`0`	*constant zero*
	`succ t`	*successor*
	`pred t`	*predecessor*
	`iszero t`	*zero test*

Terminology:

- `t` here is a *metavariable*

Abstract vs. concrete syntax

Q: Does this grammar define a set of *character strings*, a set of *token lists*, or a set of *abstract syntax trees*?

Abstract vs. concrete syntax

Q: Does this grammar define a set of *character strings*, a set of *token lists*, or a set of *abstract syntax trees*?

A: In a sense, all three. But we are primarily interested, here, in abstract syntax trees.

For this reason, grammars like the one on the previous slide are sometimes called *abstract grammars*. An abstract grammar *defines* a set of abstract syntax trees and *suggests* a mapping from character strings to trees.

We then *write* terms as linear character strings rather than trees simply for convenience. If there is any potential confusion about what tree is intended, we use parentheses to disambiguate.

Q: So, are

```
succ 0
succ (0)
(((succ (((((0)))))))))
```

"the same term"?

What about

```
succ 0
pred (succ (succ 0))
```

?

A more explicit form of the definition

The set $\mathcal{T}$ of *terms* is the smallest set such that

1. $\{\texttt{true}, \texttt{false}, \texttt{0}\} \subseteq \mathcal{T}$;
2. if $\texttt{t}_1 \in \mathcal{T}$, then $\{\texttt{succ t}_1, \texttt{pred t}_1, \texttt{iszero t}_1\} \subseteq \mathcal{T}$;
3. if $\texttt{t}_1 \in \mathcal{T}$, $\texttt{t}_2 \in \mathcal{T}$, and $\texttt{t}_3 \in \mathcal{T}$, then $\texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \in \mathcal{T}$.

Inference rules

An alternate notation for the same definition:

$$\frac{}{\texttt{true} \in \mathcal{T}} \qquad \frac{}{\texttt{false} \in \mathcal{T}} \qquad \frac{}{0 \in \mathcal{T}}$$

$$\frac{t_1 \in \mathcal{T}}{\texttt{succ}\ t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{pred}\ t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{iszero}\ t_1 \in \mathcal{T}}$$

$$\frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \in \mathcal{T}}$$

Note that "the smallest set closed under..." is implied (but often not stated explicitly).

Terminology:

- axiom vs. rule
- concrete rule vs. rule scheme

Terms, concretely

Define an infinite sequence of sets, $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, ..., as follows:

$$\begin{array}{lcl}
\mathcal{S}_0 & = & \emptyset \\
\mathcal{S}_{i+1} & = & \{\texttt{true}, \texttt{false}, 0\} \\
 & \cup & \{\texttt{succ } t_1, \texttt{pred } t_1, \texttt{iszero } t_1 \mid t_1 \in \mathcal{S}_i\} \\
 & \cup & \{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \mid t_1, t_2, t_3 \in \mathcal{S}_i\}
\end{array}$$

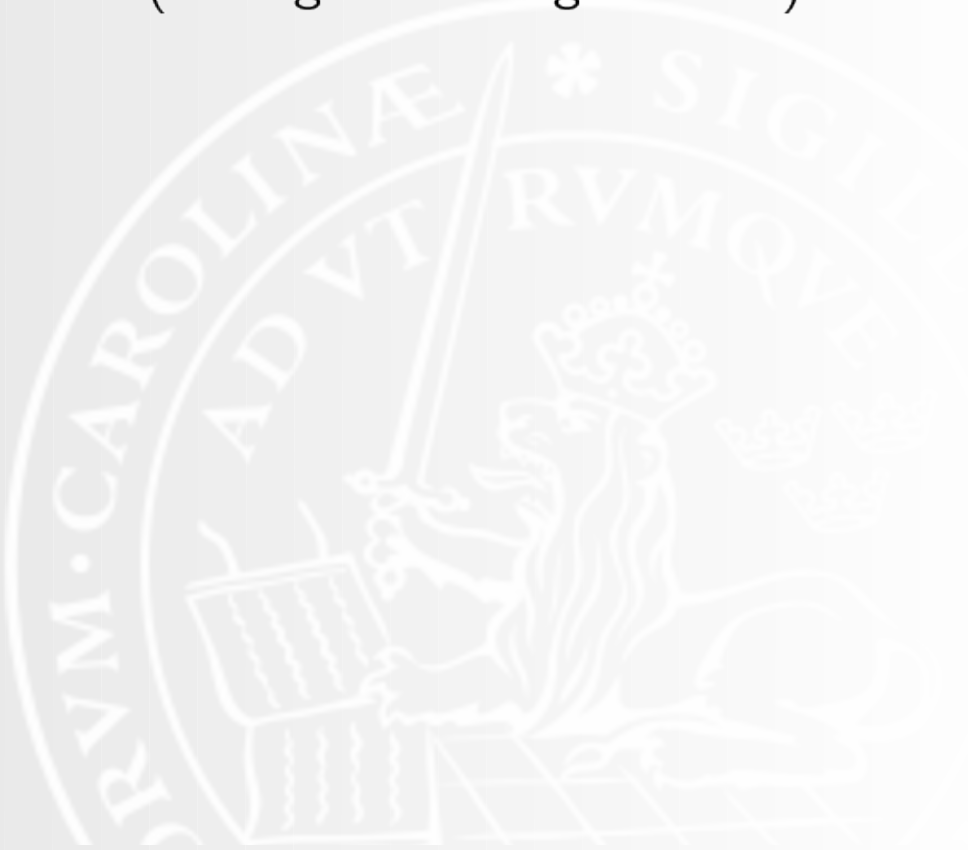
Now let

$$\mathcal{S} = \bigcup_i \mathcal{S}_i$$

Comparing the definitions

We have seen two different presentations of terms:

1. as the *smallest* set that is *closed* under certain rules ($\mathcal{T}$)
   - explicit inductive definition
   - BNF shorthand
   - inference rule shorthand
2. as the *limit* ($\mathcal{S}$) of a series of sets (of larger and larger terms)



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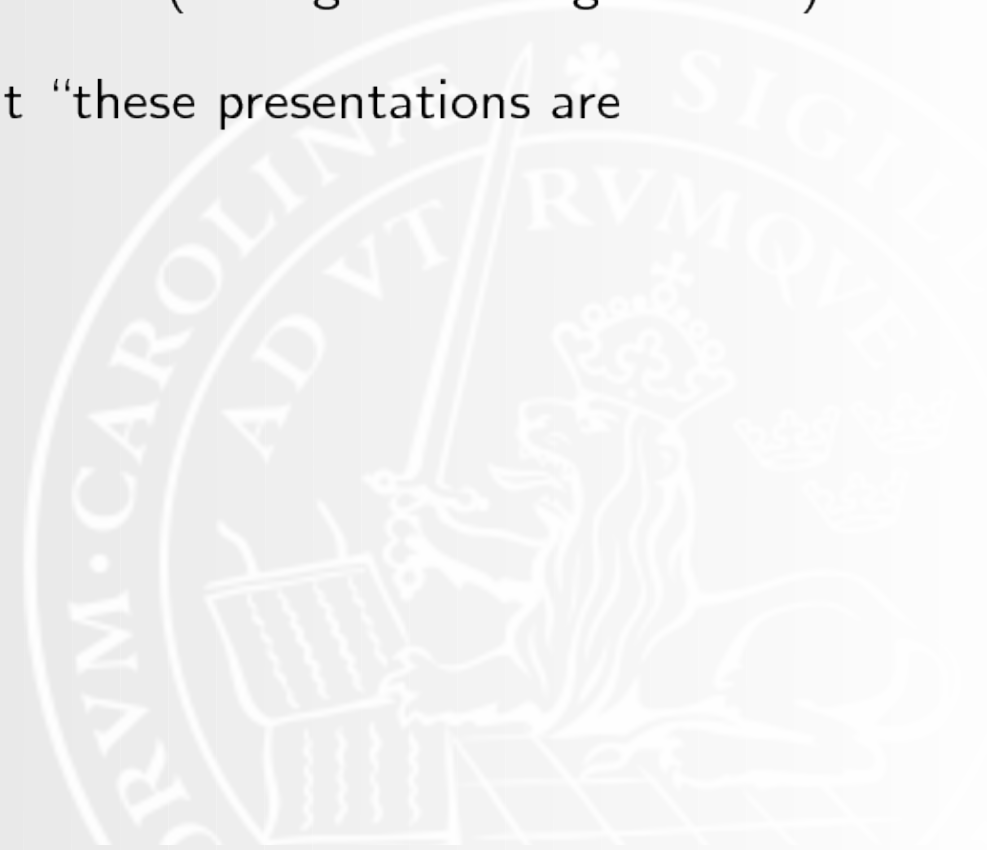
What does it mean to assert that "these presentations are equivalent"?



Why two definitions?

The two ways of defining the set of terms are both useful:

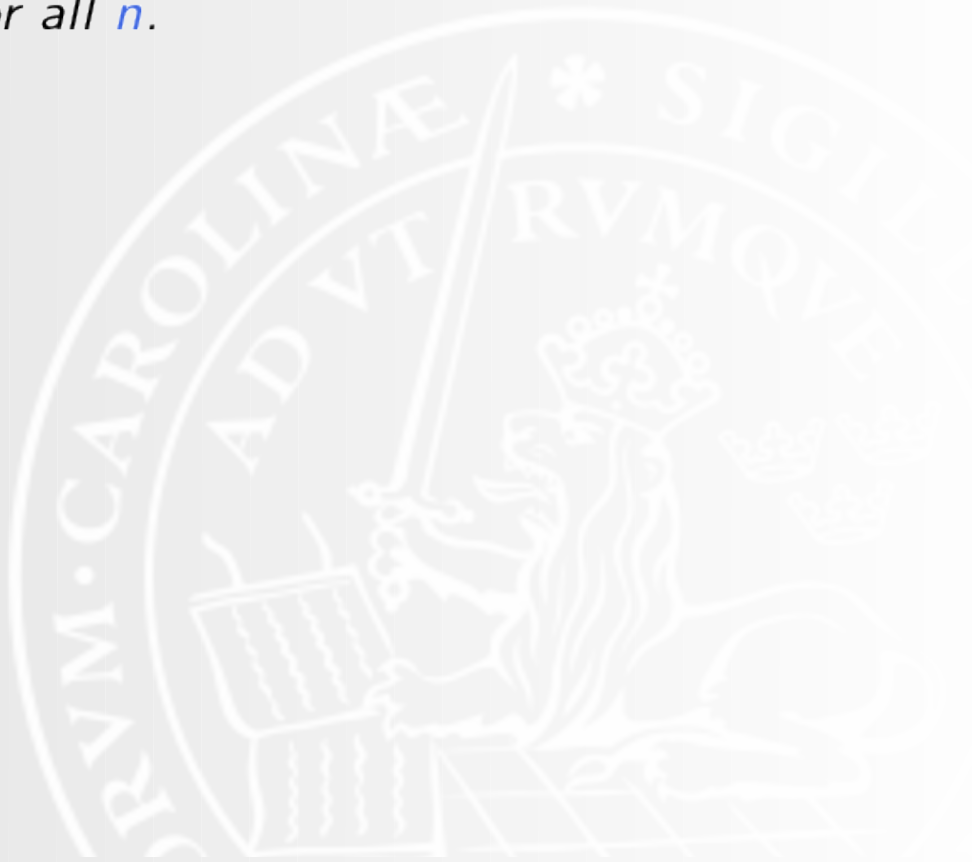
1. the definition of terms as the smallest set with a certain closure property is compact and easy to read
2. the definition of the set of terms as the limit of a sequence gives us an *induction principle* for proving things about terms...

Induction

Induction

Principle of *ordinary induction* on natural numbers:

> *Suppose that P is a predicate on the natural numbers. Then:*
>
> > *If $P(0)$*
> > *and, for all i, $P(i)$ implies $P(i+1)$,*
> > *then $P(n)$ holds for all n.*



Example

Theorem: $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$, for every n.
Proof: Let $P(i)$ be "$2^0 + 2^1 + \ldots + 2^i = 2^{i+1} - 1$."

- Show $P(0)$:

$$2^0 = 1 = 2^1 - 1$$

- Show that $P(i)$ implies $P(i+1)$:

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{i+1} &= (2^0 + 2^1 + \ldots + 2^i) + 2^{i+1} \\ &= (2^{i+1} - 1) + 2^{i+1} && \text{by IH} \\ &= 2 \cdot (2^{i+1}) - 1 \\ &= 2^{i+2} - 1 \end{aligned}$$

- The result ($P(n)$ for all n) follows by the principle of (ordinary) induction.

Shorthand form

Theorem: $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$, for every n.

Proof: By induction on n.

- Base case ($n = 0$):

$$2^0 = 1 = 2^1 - 1$$

- Inductive case ($n = i + 1$):

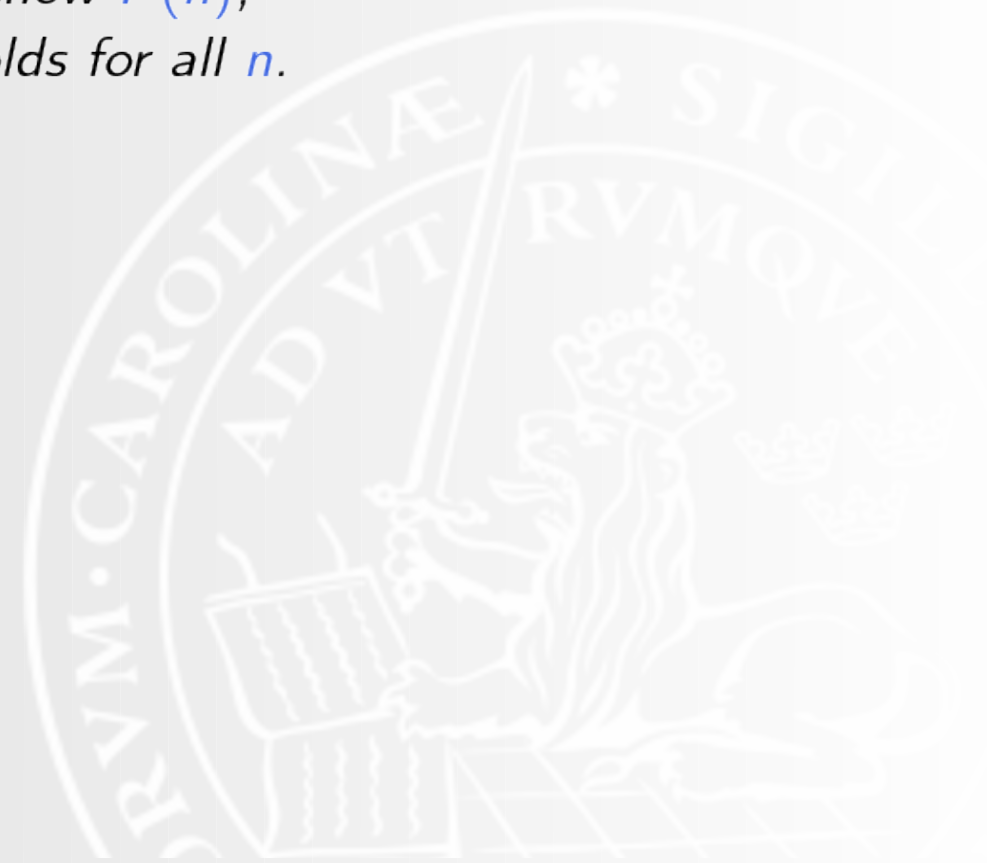
$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{i+1} &= (2^0 + 2^1 + \ldots + 2^i) + 2^{i+1} \\ &= (2^{i+1} - 1) + 2^{i+1} && \text{IH} \\ &= 2 \cdot (2^{i+1}) - 1 \\ &= 2^{i+2} - 1 \end{aligned}$$

Complete Induction

Principle of *complete induction* on natural numbers:

Suppose that P is a predicate on the natural numbers. Then:

If, for each natural number n,
given $P(i)$ for all $i < n$
we can show $P(n)$,
then $P(n)$ holds for all n.



Complete versus ordinary induction

Ordinary and complete induction are *interderivable* — assuming one, we can prove the other.

Thus, the choice of which to use for a particular proof is purely a question of style.

We'll see some other (equivalent) styles as we go along.

Induction on Syntax

Induction on Terms

Definition: The *depth* of a term t is the smallest i such that $t \in \mathcal{S}_i$.

From the definition of $\mathcal{S}$, it is clear that, if a term t is in $\mathcal{S}_i$, then all of its immediate subterms must be in $\mathcal{S}_{i-1}$, i.e., they must have strictly smaller depths.

This observation justifies the *principle of induction on terms*. Let P be a predicate on terms.

> *If, for each term s,*
> *given $P(r)$ for all immediate subterms r of s*
> *we can show $P(s)$,*
> *then $P(t)$ holds for all t.*

Inductive Function Definitions

The set of constants appearing in a term $\mathtt{t}$, written $\mathit{Consts}(\mathtt{t})$, is defined as follows:

$$
\begin{array}{lcl}
\mathit{Consts}(\mathtt{true}) & = & \{\mathtt{true}\} \\
\mathit{Consts}(\mathtt{false}) & = & \{\mathtt{false}\} \\
\mathit{Consts}(\mathtt{0}) & = & \{\mathtt{0}\} \\
\mathit{Consts}(\mathtt{succ\ t_1}) & = & \mathit{Consts}(\mathtt{t_1}) \\
\mathit{Consts}(\mathtt{pred\ t_1}) & = & \mathit{Consts}(\mathtt{t_1}) \\
\mathit{Consts}(\mathtt{iszero\ t_1}) & = & \mathit{Consts}(\mathtt{t_1}) \\
\mathit{Consts}(\mathtt{if\ t_1\ then\ t_2\ else\ t_3}) & = & \mathit{Consts}(\mathtt{t_1}) \cup \mathit{Consts}(\mathtt{t_2}) \\
 & & \cup \mathit{Consts}(\mathtt{t_3})
\end{array}
$$

Simple, right?

First question:

Normally, a "definition" just assigns a convenient name to a previously-known thing. But here, the "thing" on the right-hand side involves the very name that we are "defining"!

So in what sense is this a definition??

Second question:

Suppose we had written this instead...

The set of constants appearing in a term t, written *BadConsts*(t), is defined as follows:

$$
\begin{array}{lcl}
\mathit{BadConsts}(\texttt{true}) & = & \{\texttt{true}\} \\
\mathit{BadConsts}(\texttt{false}) & = & \{\texttt{false}\} \\
\mathit{BadConsts}(0) & = & \{0\} \\
\mathit{BadConsts}(0) & = & \{\} \\
\mathit{BadConsts}(\texttt{succ } \texttt{t}_1) & = & \mathit{BadConsts}(\texttt{t}_1) \\
\mathit{BadConsts}(\texttt{pred } \texttt{t}_1) & = & \mathit{BadConsts}(\texttt{t}_1) \\
\mathit{BadConsts}(\texttt{iszero } \texttt{t}_1) & = & \mathit{BadConsts}(\texttt{iszero (iszero } \texttt{t}_1\texttt{)})
\end{array}
$$

What is the essential difference between these two definitions?
How do we tell the difference between well-formed inductive definitions and ill-formed ones?
What, exactly, does a well-formed inductive definition mean?

What is a function?

Recall that a *function* f from A (its domain) to B (its co-domain) can be viewed as a two-place *relation* (called the "graph" of the function) with certain properties:

- It is *total*: Every element of its domain occurs at least once in its graph. More precisely:

  For every $a \in A$, *there exists some* $b \in B$ *such that* $(a, b) \in f$.

- It is *deterministic*: every element of its domain occurs at most once in its graph. More precisely:

  If $(a, b_1) \in f$ *and* $(a, b_2) \in f$, *then* $b_1 = b_2$.

We have seen how to define relations inductively. E.g....
Let *Consts* be the smallest two-place relation closed under the following rules:

$$(\texttt{true}, \{\texttt{true}\}) \in \mathit{Consts}$$

$$(\texttt{false}, \{\texttt{false}\}) \in \mathit{Consts}$$

$$(\texttt{0}, \{\texttt{0}\}) \in \mathit{Consts}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{Consts}}{(\texttt{succ } \texttt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{Consts}}{(\texttt{pred } \texttt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{Consts}}{(\texttt{iszero } \texttt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\texttt{t}_1, C_1) \in \mathit{Consts} \qquad (\texttt{t}_2, C_2) \in \mathit{Consts} \qquad (\texttt{t}_3, C_3) \in \mathit{Consts}}{(\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3, C_1 \cup C_2 \cup C_3) \in \mathit{Consts}}$$

This definition certainly defines a *relation* (i.e., the smallest one with a certain closure property).

Q: How can we be sure that this relation is a *function*?

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Q: How can we be sure that this relation is a *function*?

A: *Prove it!*

Theorem:

The relation *Consts* defined by the inference rules a couple of slides ago is total and deterministic.

I.e., for each term t there is exactly one set of terms C such that $(t, C) \in \mathit{Consts}$.

Proof:

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Proof: By induction on t.
To apply the induction principle for terms, we must show, for an arbitrary term t, that if

> for each immediate subterm s of t, there is exactly one set of terms C_s such that $(s, C_s) \in \mathit{Consts}$

then

> there is exactly one set of terms C such that $(t, C) \in \mathit{Consts}$.

Proceed by cases on the form of $\texttt{t}$.

- If $\texttt{t}$ is $\texttt{0}$, $\texttt{true}$, or $\texttt{false}$, then we can immediately see from the definition of *Consts* that there is exactly one set of terms C (namely $\{\texttt{t}\}$) such that $(\texttt{t}, C) \in \mathit{Consts}$.

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- If $\mathtt{t}$ is $\mathtt{succ}\ \mathtt{t}_1$, then the induction hypothesis tells us that there is exactly one set of terms C_1 such that $(\mathtt{t}_1, C_1) \in \mathit{Consts}$. But then it is clear from the definition of *Consts* that there is exactly one set C (namely C_1) such that $(\mathtt{t}, C) \in \mathit{Consts}$.

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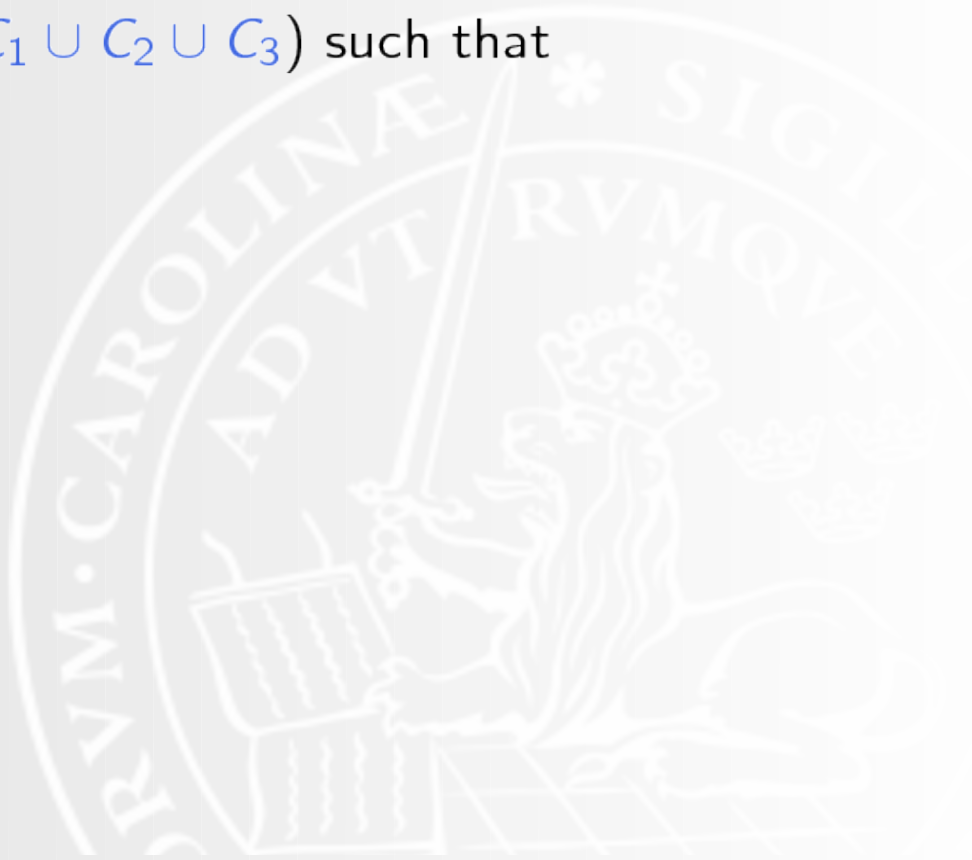
  Similarly when $\mathtt{t}$ is $\mathtt{pred}\ \mathtt{t}_1$ or $\mathtt{iszero}\ \mathtt{t}_1$.

- If $\mathtt{t}$ is $\mathtt{if\ s_1\ then\ s_2\ else\ s_3}$, then the induction hypothesis tells us
  - there is exactly one set of terms C_1 such that $(\mathtt{t}_1, C_1) \in \mathit{Consts}$
  - there is exactly one set of terms C_2 such that $(\mathtt{t}_2, C_2) \in \mathit{Consts}$
  - there is exactly one set of terms C_3 such that $(\mathtt{t}_3, C_3) \in \mathit{Consts}$

  But then it is clear from the definition of Consts that there is exactly one set C (namely $C_1 \cup C_2 \cup C_3$) such that $(\mathtt{t}, C) \in \mathit{Consts}$.



How about the bad definition?

$$(\texttt{true}, \{\texttt{true}\}) \in \mathit{BadConsts}$$

$$(\texttt{false}, \{\texttt{false}\}) \in \mathit{BadConsts}$$

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$$\frac{(\texttt{t}_1, C) \in \mathit{BadConsts}}{(\texttt{succ t}_1, C) \in \mathit{BadConsts}}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{BadConsts}}{(\texttt{pred t}_1, C) \in \mathit{BadConsts}}$$

$$\frac{(\texttt{iszero (iszero t}_1\texttt{)}, C) \in \mathit{BadConsts}}{(\texttt{iszero t}_1, C) \in \mathit{BadConsts}}$$

This set of rules defines a perfectly good *relation* — it's just that this relation does not happen to be a function!

Just for fun, let's calculate some cases of this relation...

- For what values of C do we have $(\texttt{false}, C) \in \mathit{BadConsts}$?

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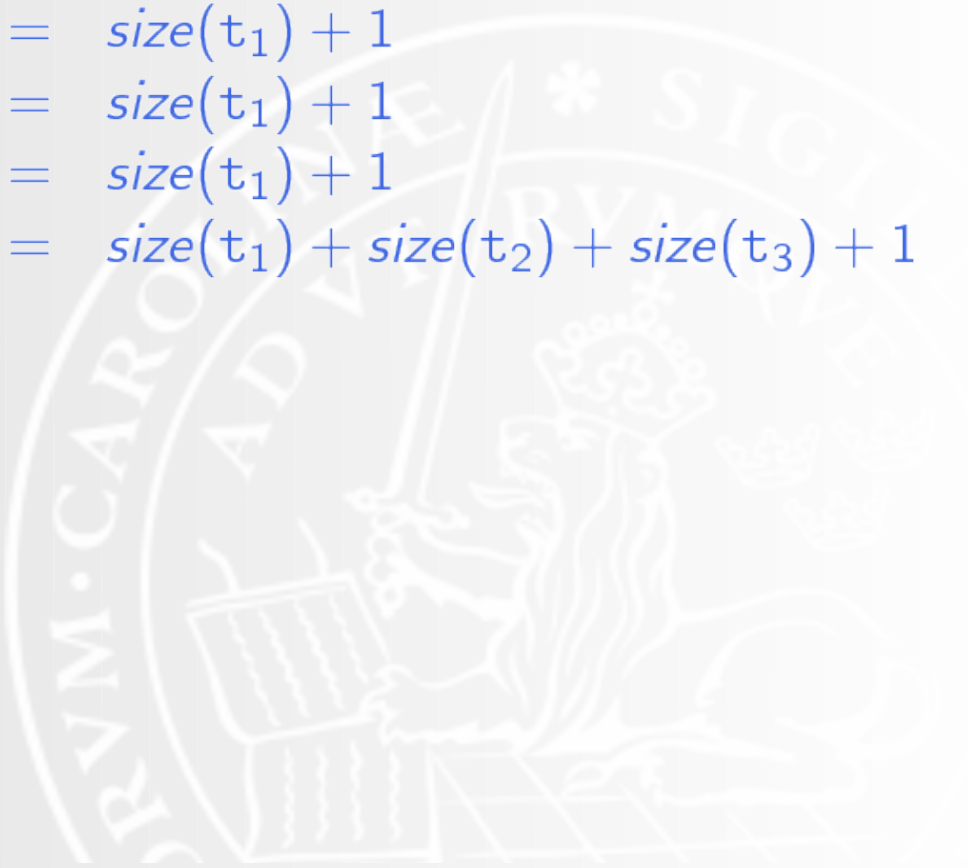
Another Inductive Definition

$$
\begin{array}{lcl}
\mathit{size}(\texttt{true}) & = & 1 \\
\mathit{size}(\texttt{false}) & = & 1 \\
\mathit{size}(\texttt{0}) & = & 1 \\
\mathit{size}(\texttt{succ } \texttt{t}_1) & = & \mathit{size}(\texttt{t}_1) + 1 \\
\mathit{size}(\texttt{pred } \texttt{t}_1) & = & \mathit{size}(\texttt{t}_1) + 1 \\
\mathit{size}(\texttt{iszero } \texttt{t}_1) & = & \mathit{size}(\texttt{t}_1) + 1 \\
\mathit{size}(\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3) & = & \mathit{size}(\texttt{t}_1) + \mathit{size}(\texttt{t}_2) + \mathit{size}(\texttt{t}_3) + 1
\end{array}
$$



Another proof by induction

Theorem: The number of distinct constants in a term is at most the size of the term. I.e., $|\mathit{Consts}(\mathtt{t})| \leq \mathit{size}(\mathtt{t})$.

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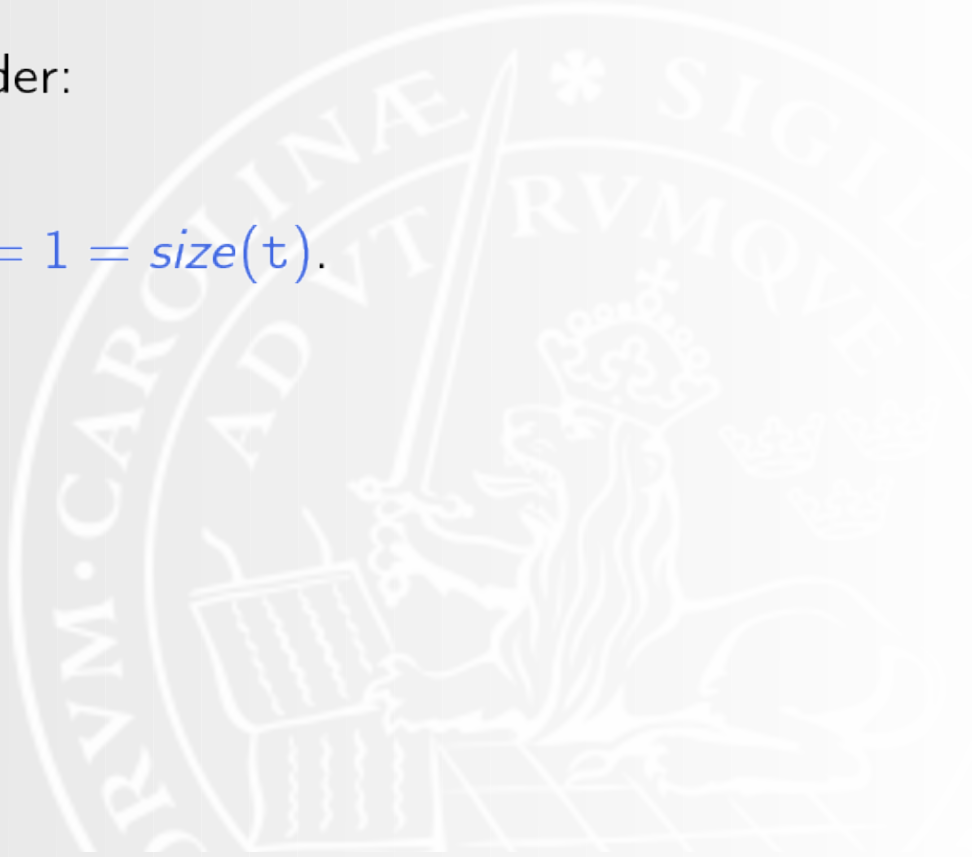
There are "three" cases to consider:

Case: $\mathtt{t}$ is a constant

Immediate: $|\mathit{Consts}(\mathtt{t})| = |\{\mathtt{t}\}| = 1 = \mathit{size}(\mathtt{t})$.



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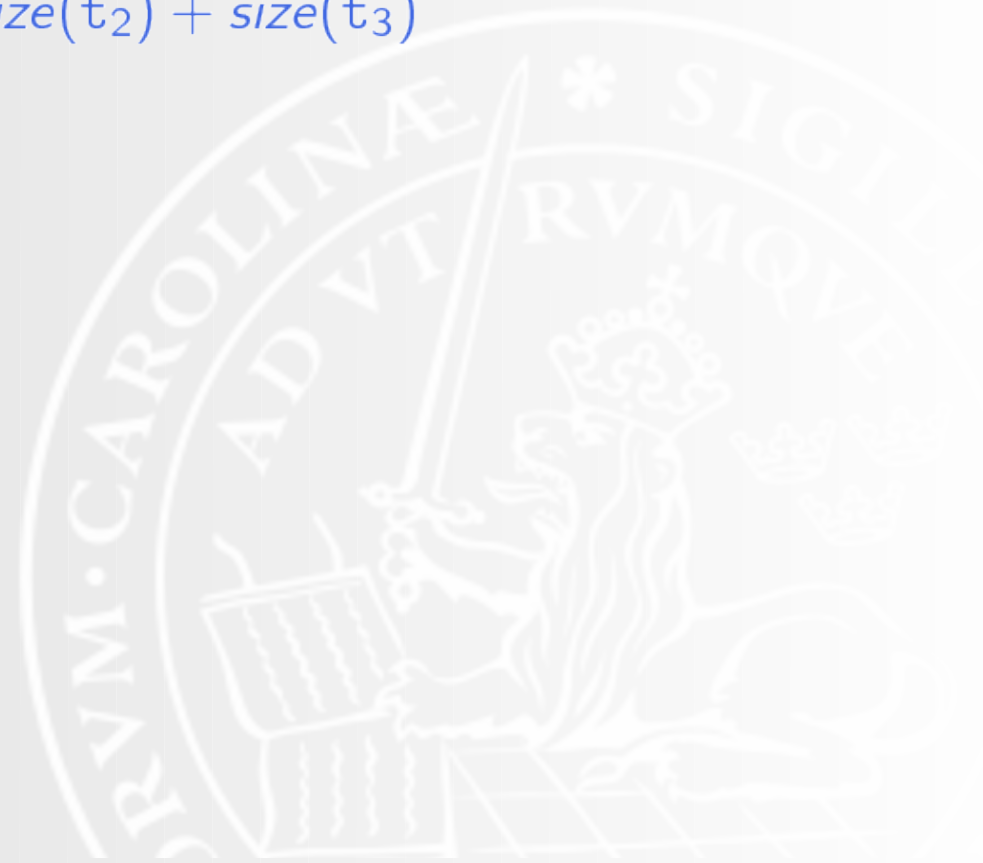
Case: $\mathtt{t} = \mathtt{succ}\ \mathtt{t}_1$, $\mathtt{pred}\ \mathtt{t}_1$, or $\mathtt{iszero}\ \mathtt{t}_1$

By the induction hypothesis, $|\mathit{Consts}(\mathtt{t}_1)| \leq \mathit{size}(\mathtt{t}_1)$. We now calculate as follows:
$|\mathit{Consts}(\mathtt{t})| = |\mathit{Consts}(\mathtt{t}_1)| \leq \mathit{size}(\mathtt{t}_1) < \mathit{size}(\mathtt{t})$.

Case: $\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$

By the induction hypothesis, $|\mathit{Consts}(\mathtt{t}_1)| \leq \mathit{size}(\mathtt{t}_1)$, $|\mathit{Consts}(\mathtt{t}_2)| \leq \mathit{size}(\mathtt{t}_2)$, and $|\mathit{Consts}(\mathtt{t}_3)| \leq \mathit{size}(\mathtt{t}_3)$. We now calculate as follows:

$$\begin{aligned} |\mathit{Consts}(\mathtt{t})| &= |\mathit{Consts}(\mathtt{t}_1) \cup \mathit{Consts}(\mathtt{t}_2) \cup \mathit{Consts}(\mathtt{t}_3)| \\ &\leq |\mathit{Consts}(\mathtt{t}_1)| + |\mathit{Consts}(\mathtt{t}_2)| + |\mathit{Consts}(\mathtt{t}_3)| \\ &\leq \mathit{size}(\mathtt{t}_1) + \mathit{size}(\mathtt{t}_2) + \mathit{size}(\mathtt{t}_3) \\ &< \mathit{size}(\mathtt{t}). \end{aligned}$$



Operational Semantics (Evaluation)

Abstract Machines

An *abstract machine* consists of:

- a set of *states*
- a *transition relation* on states, written $\longrightarrow$

We read "$t \longrightarrow t'$" as "t evaluates to t' in one step".

A state records *all* the information in the machine at a given moment. For example, an abstract-machine-style description of a conventional microprocessor would include the program counter, the contents of the registers, the contents of main memory, and the machine code program being executed.

Abstract Machines

For the very simple languages we are considering at the moment, however, the term being evaluated is the whole state of the abstract machine.

Nb. Often, the transition relation is actually a partial function: i.e., from a given state, there is at most one possible next state. But in general there may be many.

Operational semantics for Booleans

Syntax of terms and values

t ::=		*terms*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
v ::=		*values*
	`true`	*true value*
	`false`	*false value*

Evaluation relation for Booleans

The evaluation relation $t \longrightarrow t'$ is the smallest relation closed under the following rules:

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \quad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \quad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \quad \text{(E-If)}$$

Terminology

Computation rules:

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \quad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \quad \text{(E-IfFalse)}$$

Congruence rule:

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \quad \text{(E-If)}$$

Computation rules perform "real" computation steps.
Congruence rules determine *where* computation rules can be applied next.

Evaluation, more explicitly

$\longrightarrow$ is the smallest two-place relation closed under the following rules:

$$((\texttt{if true then } t_2 \texttt{ else } t_3), t_2) \in \longrightarrow$$

$$((\texttt{if false then } t_2 \texttt{ else } t_3), t_3) \in \longrightarrow$$

$$\frac{(t_1, t_1') \in \longrightarrow}{((\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3), (\texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3)) \in \longrightarrow}$$

The notation $t \longrightarrow t'$ is short-hand for $(t, t') \in \longrightarrow$.

Derivations

We can record the "justification" for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- These trees are called *derivation trees* (or just *derivations*).
- The final statement in a derivation is its *conclusion*.
- We say that the derivation is a *witness* for its conclusion (or a *proof* of its conclusion) — it records all the reasoning steps that justify the conclusion.

Observation

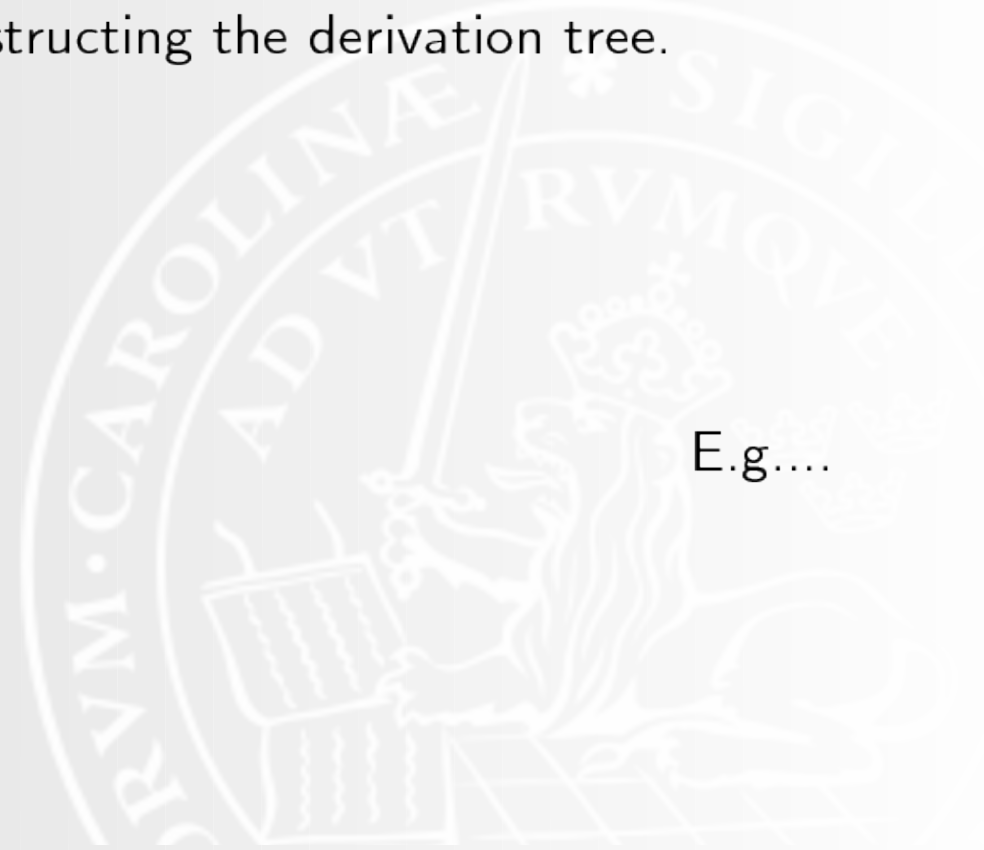
Lemma: Suppose we are given a derivation tree $\mathcal{D}$ witnessing the pair (t, t') in the evaluation relation. Then either

1. the final rule used in $\mathcal{D}$ is E-IfTrue and we have $t = \texttt{if true then } t_2 \texttt{ else } t_3$ and $t' = t_2$, for some t_2 and t_3, or
2. the final rule used in $\mathcal{D}$ is E-IfFalse and we have $t = \texttt{if false then } t_2 \texttt{ else } t_3$ and $t' = t_3$, for some t_2 and t_3, or
3. the final rule used in $\mathcal{D}$ is E-If and we have $t = \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3$ and $t' = \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3$, for some t_1, t_1', t_2, and t_3; moreover, the immediate subderivation of $\mathcal{D}$ witnesses $(t_1, t_1') \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation "by induction on derivation trees."

Given an arbitrary derivation $\mathcal{D}$ with conclusion $\mathtt{t} \longrightarrow \mathtt{t}'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.



E.g....

Induction on Derivations — Example

Theorem: If $t \longrightarrow t'$, i.e., if $(t, t') \in \longrightarrow$, then $\mathit{size}(t) > \mathit{size}(t')$.
Proof: By induction on a derivation $\mathcal{D}$ of $t \longrightarrow t'$.

1. Suppose the final rule used in $\mathcal{D}$ is E-IfTrue, with $t = \texttt{if true then } t_2 \texttt{ else } t_3$ and $t' = t_2$. Then the result is immediate from the definition of *size*.
2. Suppose the final rule used in $\mathcal{D}$ is E-IfFalse, with $t = \texttt{if false then } t_2 \texttt{ else } t_3$ and $t' = t_3$. Then the result is again immediate from the definition of *size*.
3. Suppose the final rule used in $\mathcal{D}$ is E-If, with $t = \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3$ and $t' = \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3$, where $(t_1, t_1') \in \longrightarrow$ is witnessed by a derivation $\mathcal{D}_1$. By the induction hypothesis, $\mathit{size}(t_1) > \mathit{size}(t_1')$. But then, by the definition of *size*, we have $\mathit{size}(t) > \mathit{size}(t')$.

Values & Normal Forms

Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term t is a normal form (or "is in normal form") if there is no t' such that $t \longrightarrow t'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a "result" of evaluation.

Normal forms

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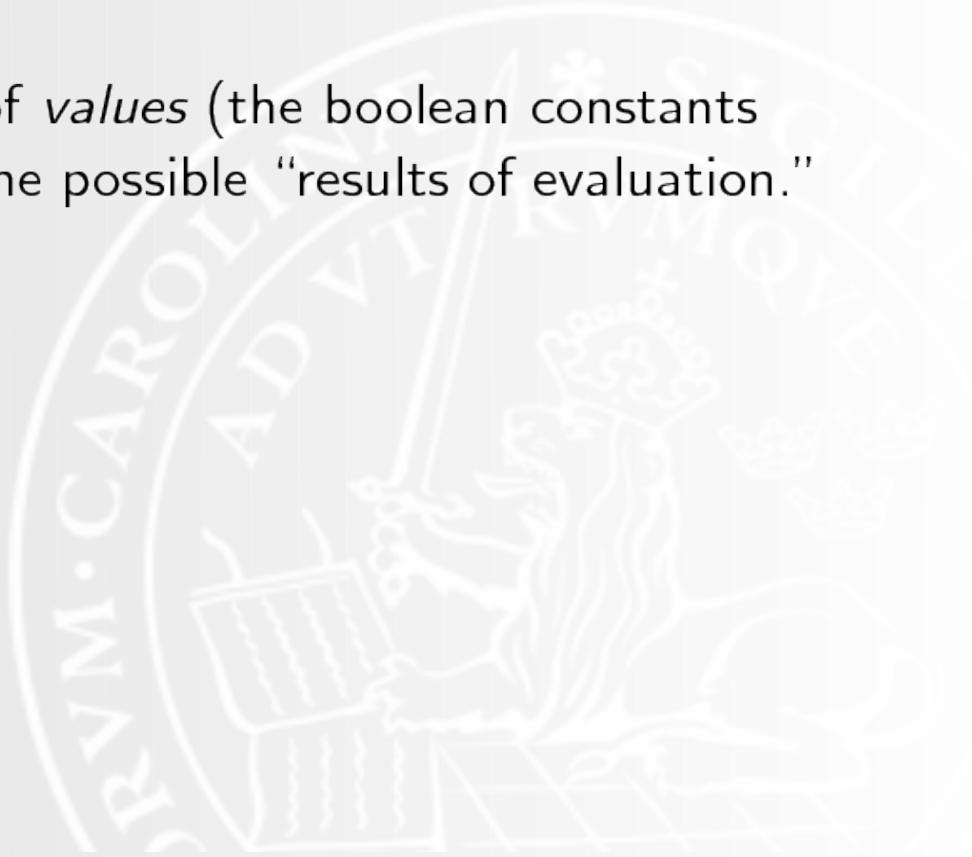
A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a "result" of evaluation.

Recall that we intended the set of *values* (the boolean constants `true` and `false`) to be exactly the possible "results of evaluation." Did we get this definition right?



Values = normal forms

Theorem: A term `t` is a value iff it is in normal form.

Proof:

The $\Longrightarrow$ direction is immediate from the definition of the evaluation relation.

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For the $\Longleftarrow$ direction, it is convenient to prove the contrapositive: If t is *not* a value, then it is *not* a normal form.

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The $\Longrightarrow$ direction is immediate from the definition of the evaluation relation.

For the $\Longleftarrow$ direction, it is convenient to prove the contrapositive: If $\mathtt{t}$ is *not* a value, then it is *not* a normal form. The argument goes by induction on $\mathtt{t}$.

Note, first, that $\mathtt{t}$ must have the form $\mathtt{if\ t_1\ then\ t_2\ else\ t_3}$ (otherwise it would be a value). If $\mathtt{t_1}$ is $\mathtt{true}$ or $\mathtt{false}$, then rule E-IfTrue or E-IfFalse applies to $\mathtt{t}$, and we are done.

Otherwise, $\mathtt{t_1}$ is not a value and so, by the induction hypothesis, there is some $\mathtt{t_1'}$ such that $\mathtt{t_1} \longrightarrow \mathtt{t_1'}$. But then rule E-If yields

$$\mathtt{if\ t_1\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}$$

i.e., $\mathtt{t}$ is not in normal form.

Numbers

New syntactic forms

`t`	::=	...	*terms*
		`0`	*constant zero*
		`succ t`	*successor*
		`pred t`	*predecessor*
		`iszero t`	*zero test*
`v`	::=	...	*values*
		`nv`	*numeric value*
`nv`	::=		*numeric values*
		`0`	*zero value*
		`succ nv`	*successor value*

New evaluation rules

$\boxed{t \longrightarrow t'}$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{succ } t_1 \longrightarrow \texttt{succ } t_1'} \quad \text{(E-Succ)}$$

$$\texttt{pred } 0 \longrightarrow 0 \quad \text{(E-PredZero)}$$

$$\texttt{pred (succ } nv_1\texttt{)} \longrightarrow nv_1 \quad \text{(E-PredSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{pred } t_1 \longrightarrow \texttt{pred } t_1'} \quad \text{(E-Pred)}$$

$$\texttt{iszero } 0 \longrightarrow \texttt{true} \quad \text{(E-IszeroZero)}$$

$$\texttt{iszero (succ } nv_1\texttt{)} \longrightarrow \texttt{false} \quad \text{(E-IszeroSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{iszero } t_1 \longrightarrow \texttt{iszero } t_1'} \quad \text{(E-IsZero)}$$

Values are normal forms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

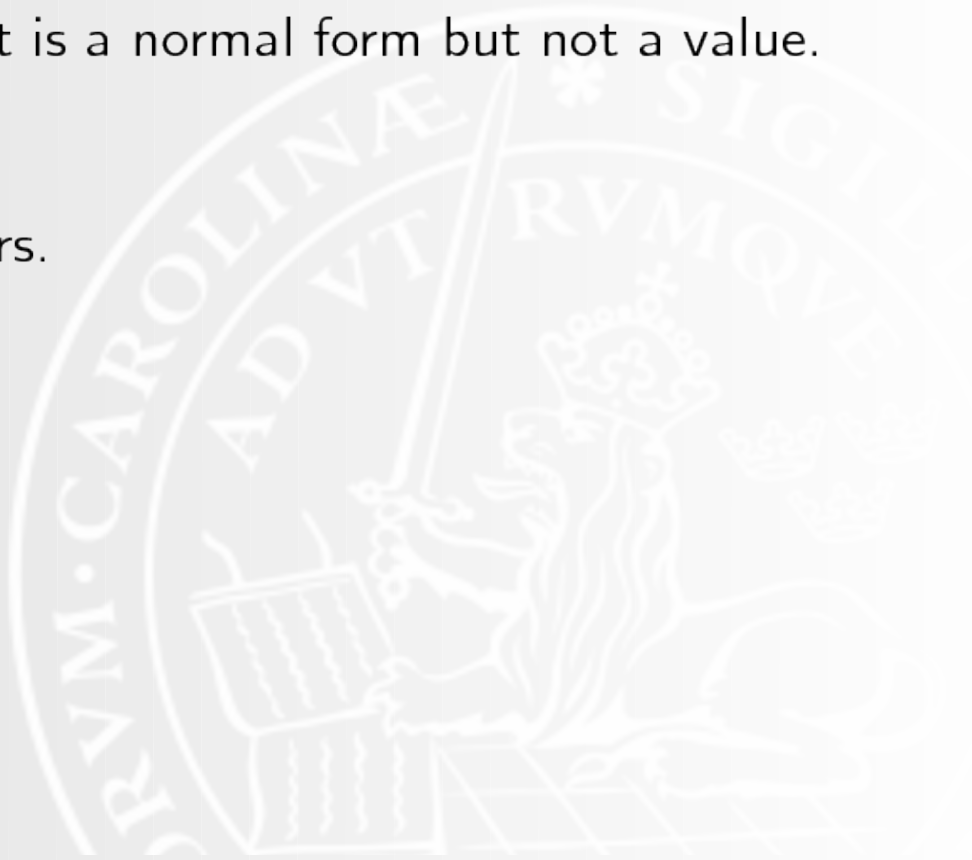
Values are normal forms, but we have stuck terms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?
No: some terms are *stuck*.

Formally, a stuck term is one that is a normal form but not a value.
What are some examples?

Stuck terms model run-time errors.



Multi-step evaluation.

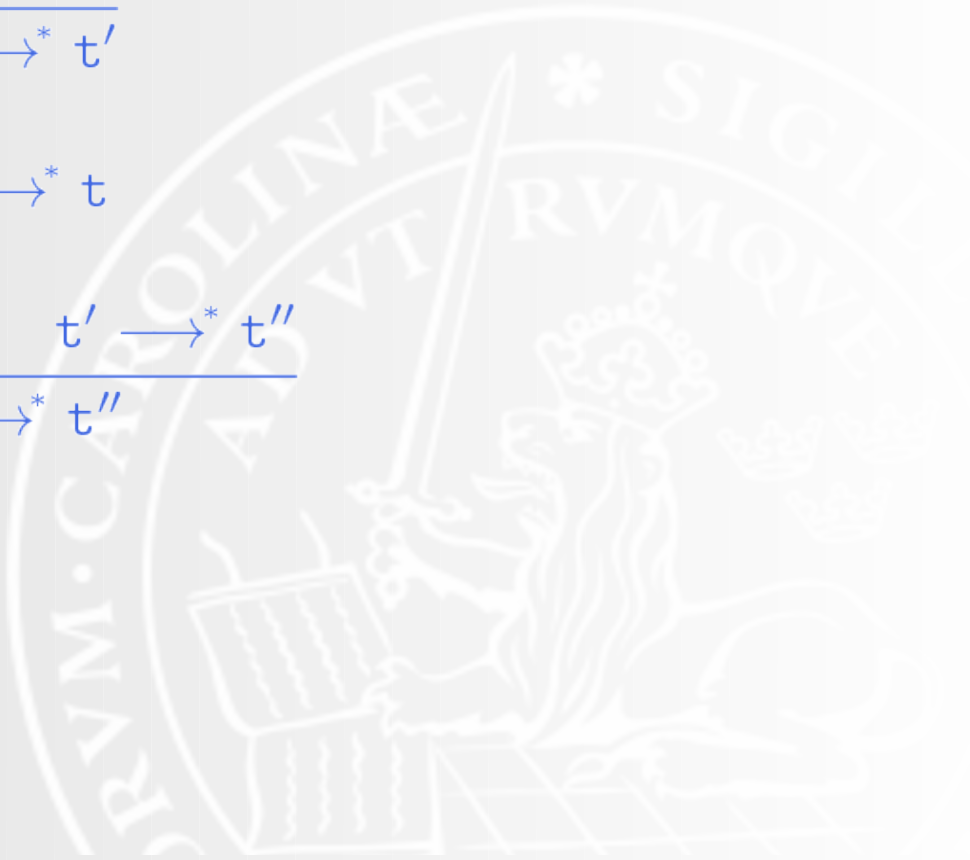
The *multi-step evaluation* relation, $\longrightarrow^*$, is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{t \longrightarrow t'}{t \longrightarrow^* t'}$$

$$t \longrightarrow^* t$$

$$\frac{t \longrightarrow^* t' \qquad t' \longrightarrow^* t''}{t \longrightarrow^* t''}$$



Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

Termination of evaluation

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Proof:

- First, recall that single-step evaluation strictly reduces the size of the term:

  $$\textit{if } t \longrightarrow t', \textit{ then } \textit{size}(t) > \textit{size}(t')$$

- Now, assume (for a contradiction) that

  $$t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ \ldots$$

  is an infinite-length sequence such that

  $$t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \cdots.$$

- Then

  $$\textit{size}(t_0) > \textit{size}(t_1) > \textit{size}(t_2) > \textit{size}(t_3) > \ldots$$

- But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

Theorem: *The relation $R \subseteq X \times X$ is terminating — i.e., there are no infinite sequences x_0, x_1, x_2, etc. such that $(x_i, x_{i+1}) \in R$ for each i.*

Proof:

1. *Choose*
   - *a well-founded set $(W, <)$ — i.e., a set W with a partial order $<$ such that there are no infinite descending chains $w_0 > w_1 > w_2 > \ldots$ in W*
   - *a function f from X to W*
2. *Show $f(x) > f(y)$ for all $(x, y) \in R$*
3. *Conclude that there are no infinite sequences x_0, x_1, x_2, etc. such that $(x_i, x_{i+1}) \in R$ for each i, since, if there were, we could construct an infinite descending chain in W.*

λ Calculus

The lambda-calculus

- If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
  - Turing complete
  - higher order (functions as data)
- Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- The *e. coli* of programming language research
- The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

```
plus3 x  =  succ (succ (succ x))
```

That is, "`plus3 x` is `succ (succ (succ x))`."

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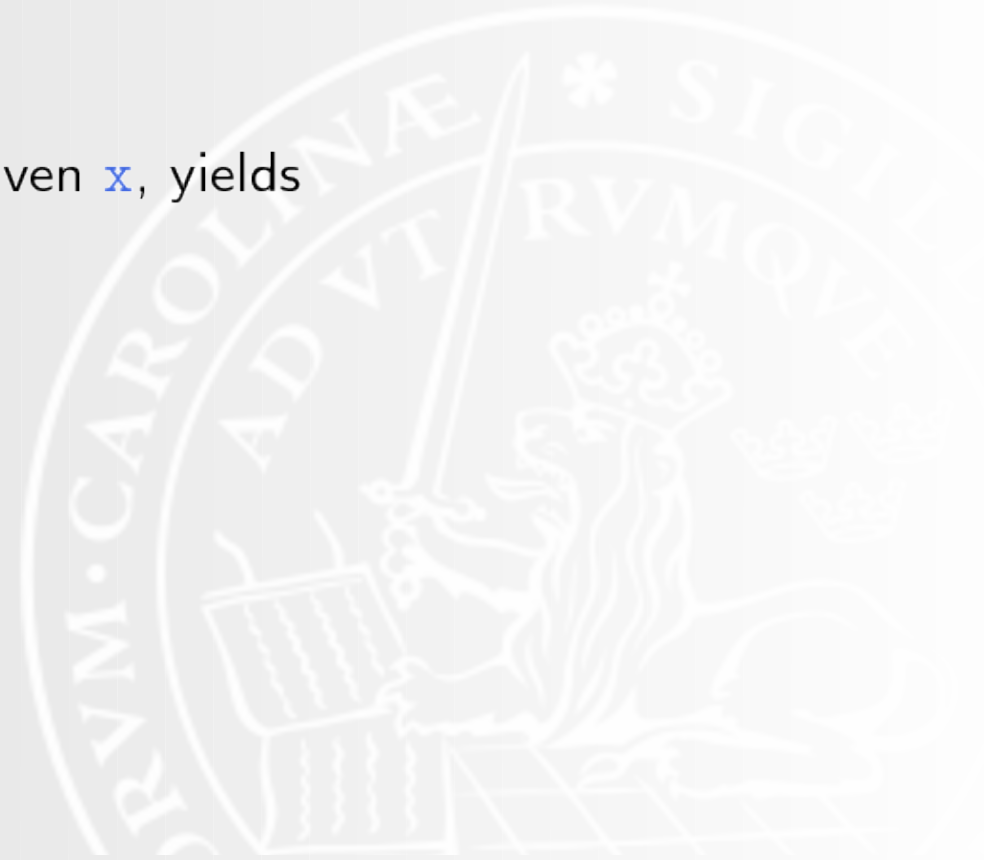
A: `plus3` is the function that, given `x`, yields `succ (succ (succ x))`.



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$$\texttt{plus3 x} \quad = \quad \texttt{succ (succ (succ x))}$$

That is, "`plus3 x` is `succ (succ (succ x))`."

Q: What is `plus3` itself?

A: `plus3` is the function that, given `x`, yields `succ (succ (succ x))`.

$$\texttt{plus3} \quad = \quad \lambda\texttt{x. succ (succ (succ x))}$$

This function exists independent of the name `plus3`.

λ`x. t` is written "`fun x` $\rightarrow$ `t`" in OCaml and "`x` $\Rightarrow$ `t`" in Scala.

So `plus3 (succ 0)` is just a convenient shorthand for "the function that, given `x`, yields `succ (succ (succ x))`, applied to `succ 0`."

```
plus3 (succ 0)
=
(λx. succ (succ (succ x))) (succ 0)
```

Abstractions over Functions

Consider the λ-abstraction

$$\texttt{g} \quad = \quad \lambda\texttt{f. f (f (succ 0))}$$

Note that the parameter variable `f` is used in the *function* position in the body of `g`. Terms like `g` are called *higher-order* functions. If we apply `g` to an argument like `plus3`, the "substitution rule" yields a nontrivial computation:

```
g plus3
  =    (λf. f (f (succ 0))) (λx. succ (succ (succ x)))
  i.e. (λx. succ (succ (succ x)))
         ((λx. succ (succ (succ x))) (succ 0))
  i.e. (λx. succ (succ (succ x)))
         (succ (succ (succ (succ 0))))
  i.e. succ (succ (succ (succ (succ (succ (succ 0))))))
```

Abstractions Returning Functions

Consider the following variant of `g`:

$$\texttt{double} \quad = \quad \lambda\texttt{f}.\ \lambda\texttt{y}.\ \texttt{f}\ (\texttt{f}\ \texttt{y})$$

I.e., `double` is the function that, when applied to a function `f`, yields a *function* that, when applied to an argument `y`, yields `f (f y)`.

Example

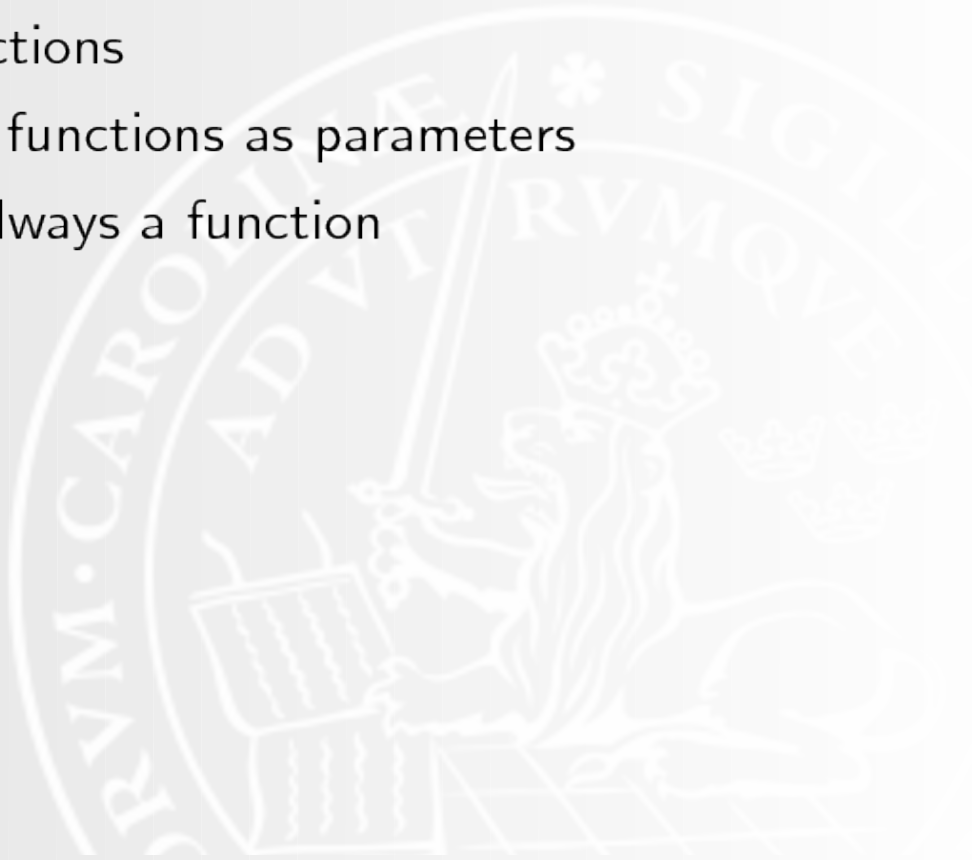
```
     double plus3 0
=    (λf. λy. f (f y))
       (λx. succ (succ (succ x)))
       0
i.e. (λy. (λx. succ (succ (succ x)))
             ((λx. succ (succ (succ x))) y))
       0
i.e. (λx. succ (succ (succ x)))
             ((λx. succ (succ (succ x))) 0)
i.e. (λx. succ (succ (succ x)))
             (succ (succ (succ 0)))
i.e. succ (succ (succ (succ (succ (succ 0)))))
```

The Pure Lambda-Calculus

As the preceding examples suggest, once we have λ-abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the "pure lambda-calculus" — *everything* is a function.

- Variables always denote functions
- Functions always take other functions as parameters
- The result of a function is always a function



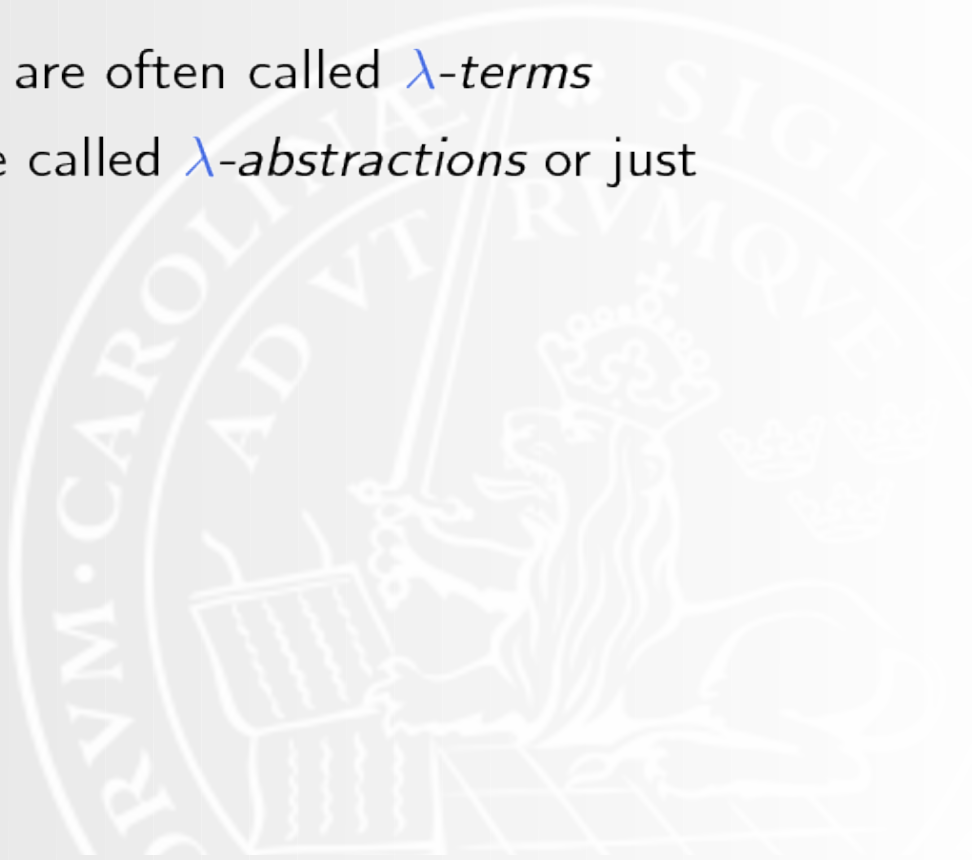
λ Calculus: Formalities

Syntax

`t` ::=		*terms*
	`x`	*variable*
	`λx.t`	*abstraction*
	`t t`	*application*

*Term*inology:

- terms in the pure λ-calculus are often called *λ-terms*
- terms of the form `λx. t` are called *λ-abstractions* or just *abstractions*



Syntactic conventions

Since λ-calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- Application associates to the left

  E.g., $t\ u\ v$ *means* $(t\ u)\ v$, *not* $t\ (u\ v)$

- Bodies of λ- abstractions extend as far to the right as possible

  E.g., $\lambda x.\ \lambda y.\ x\ y$ *means* $\lambda x.\ (\lambda y.\ x\ y)$, *not* $\lambda x.\ (\lambda y.\ x)\ y$

Scope

The λ-abstraction term $\lambda x.t$ *binds* the variable x.

The *scope* of this binding is the *body* t.

Occurrences of x inside t are said to be *bound* by the abstraction.

Occurrences of x that are *not* within the scope of an abstraction binding x are said to be *free*.

Test:

$$\lambda x.\ \lambda y.\ x\ y\ z$$

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$$\lambda x.\ \lambda y.\ x\ y\ z$$
$$\lambda x.\ (\lambda y.\ z\ y)\ y$$

Values

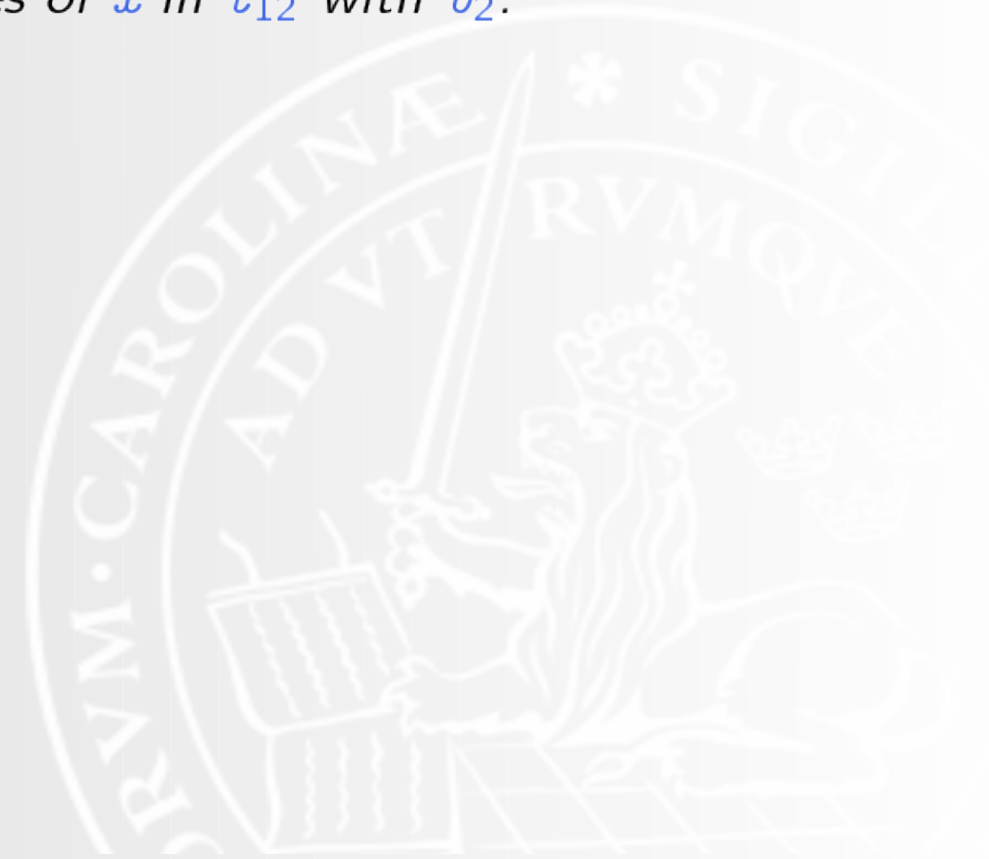
v	::=	*values*
	λx.t	*abstraction value*

Operational Semantics

Computation rule:

$$(\lambda x.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \qquad \text{(E-AppAbs)}$$

Notation: $[x \mapsto v_2]t_{12}$ *is "the term that results from substituting free occurrences of* x *in* t_{12} *with* v_2*."*



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Congruence rules:

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \qquad \text{(E-App2)}$$

Terminology

A term of the form `(λx.t) v` — that is, a λ-abstraction applied to a *value* — is called a *redex* (short for "reducible expression").

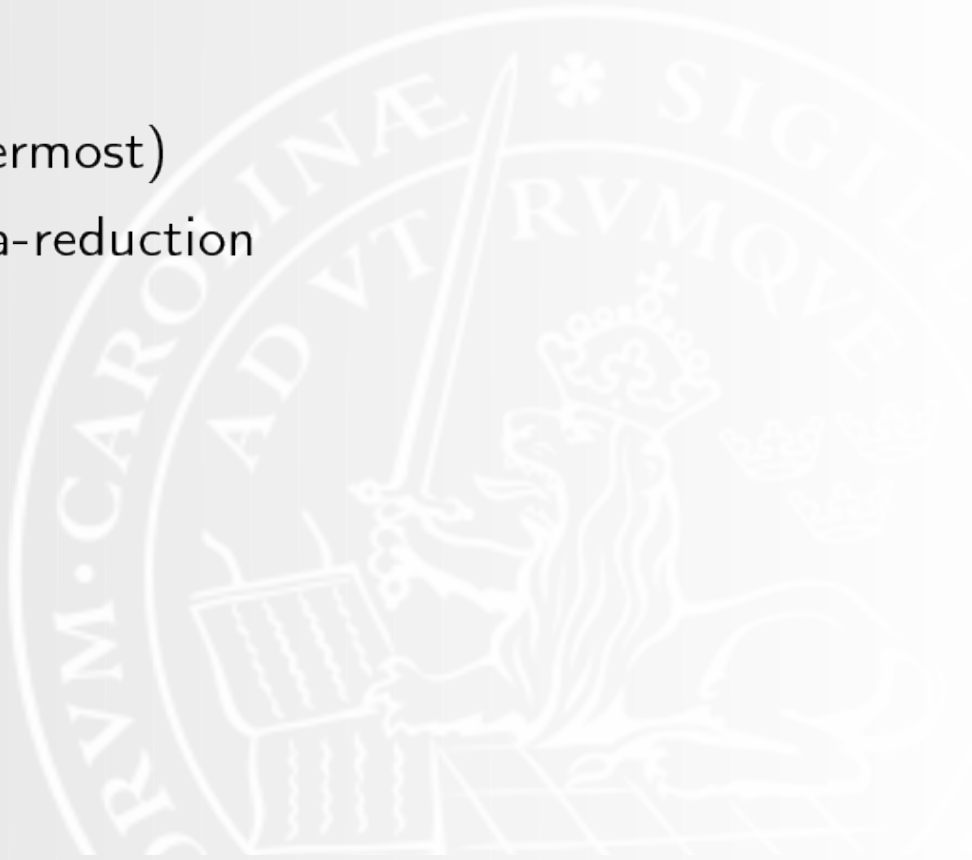
Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*.

The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages.

Some other common ones:

- Call by name (cf. Haskell)
- Normal order (leftmost/outermost)
- Full (non-deterministic) beta-reduction



Classical λ Calculus

Full beta reduction

The classical lambda calculus allows full beta reduction.

- The argument of a β-reduction to be an arbitrary term, not just a value.
- Reduction may appear anywhere in a term.

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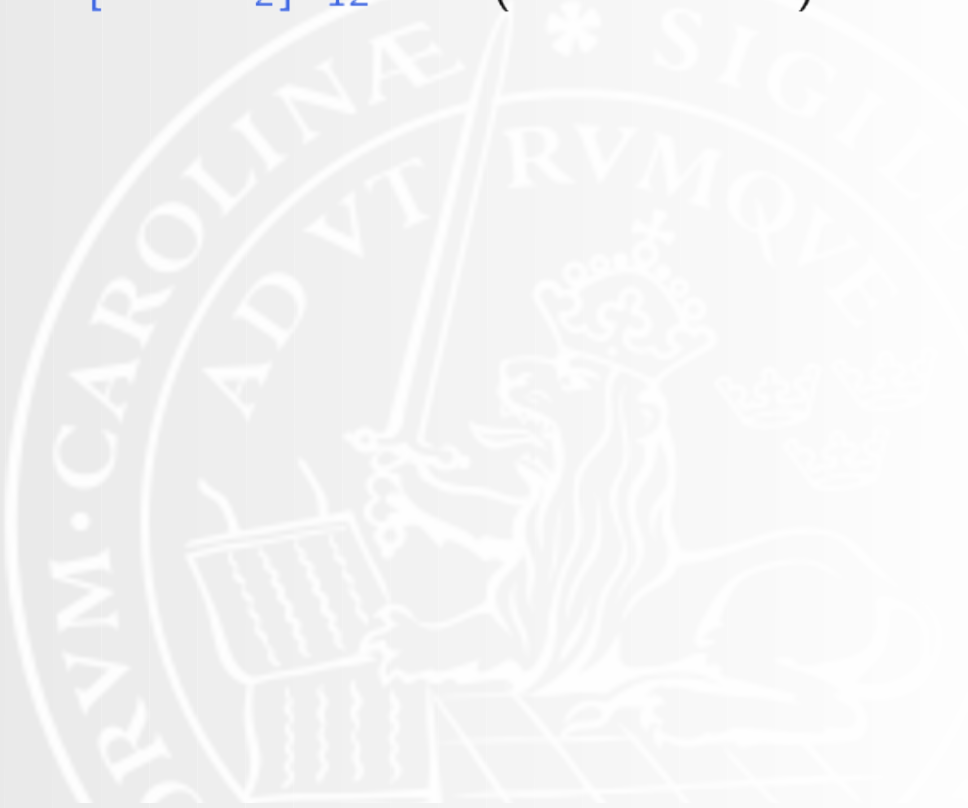

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Computation rule:

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Congruence rules:

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{t_1\ t_2 \longrightarrow t_1\ t_2'} \qquad \text{(E-App2)}$$

$$\frac{t \longrightarrow t'}{\lambda x.t \longrightarrow \lambda x.t'} \qquad \text{(E-Abs)}$$

Substitution revisited

Remember: $[x \mapsto v_2]\, t_{12}$ *is "the term that results from substituting free occurrences of* x *in* t_{12} *with* v_2*."*

This is trickier than it looks!

For example:

$$
\begin{array}{rl}
 & (\lambda \mathtt{x}.\ (\lambda \mathtt{y}.\ \mathtt{x}))\ \mathtt{y} \\
\longrightarrow & [\mathtt{x} \mapsto \mathtt{y}]\lambda \mathtt{y}.\ \mathtt{x} \\
= & ???
\end{array}
$$

Substitution revisited

Remember: $[x \mapsto v_2]t_{12}$ *is "the term that results from substituting free occurrences of* x *in* t_{12} *with* v_2*."*

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Solution:
need to rename bound variables before performing the substitution.

$$\begin{array}{rl} & (\lambda \mathtt{x}.\ (\lambda \mathtt{y}.\ \mathtt{x}))\ \mathtt{y} \\ = & (\lambda \mathtt{x}.\ (\lambda \mathtt{z}.\ \mathtt{x}))\ \mathtt{y} \\ \longrightarrow & [\mathtt{x} \mapsto \mathtt{y}]\lambda \mathtt{z}.\ \mathtt{x} \\ = & \lambda \mathtt{z}.\ \mathtt{y} \end{array}$$

Alpha conversion

Renaming bound variables is formalized as α-conversion.
Conversion rule:

$$\frac{\mathtt{y} \notin \mathtt{fv(t)}}{\lambda \mathtt{x}.\ \mathtt{t} =_\alpha \lambda \mathtt{y}.[\mathtt{x} \mapsto \mathtt{y}]\mathtt{t}} \quad (\alpha)$$

Equivalence rules:

$$\frac{\mathtt{t}_1 =_\alpha \mathtt{t}_2}{\mathtt{t}_2 =_\alpha \mathtt{t}_1} \quad (\alpha\text{-Symm})$$

$$\frac{\mathtt{t}_1 =_\alpha \mathtt{t}_2 \qquad \mathtt{t}_2 =_\alpha \mathtt{t}_3}{\mathtt{t}_1 =_\alpha \mathtt{t}_3} \quad (\alpha\text{-Trans})$$

Congruence rules: the usual ones.

Confluence

Full β-reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

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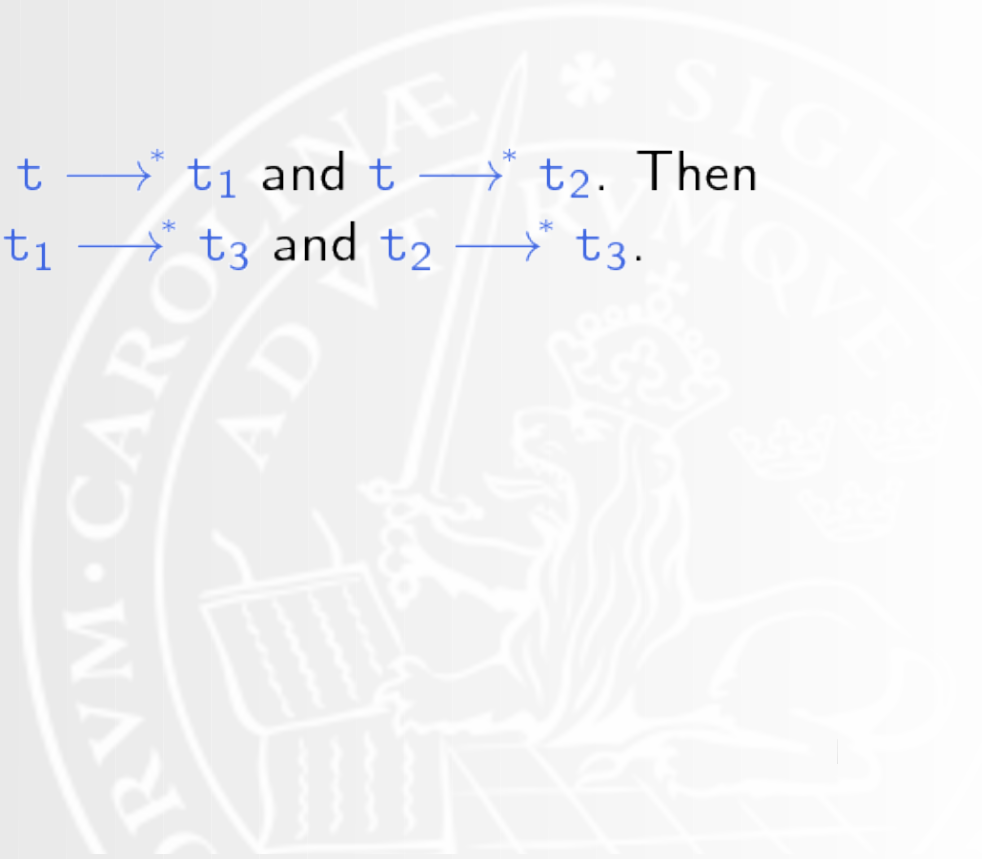
Full β-reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

The answer is no; this is a consequence of the following

Theorem [Church-Rosser]
Let t, t_1, t_2 be terms such that $t \longrightarrow^* t_1$ and $t \longrightarrow^* t_2$. Then there exists a term t_3 such that $t_1 \longrightarrow^* t_3$ and $t_2 \longrightarrow^* t_3$.

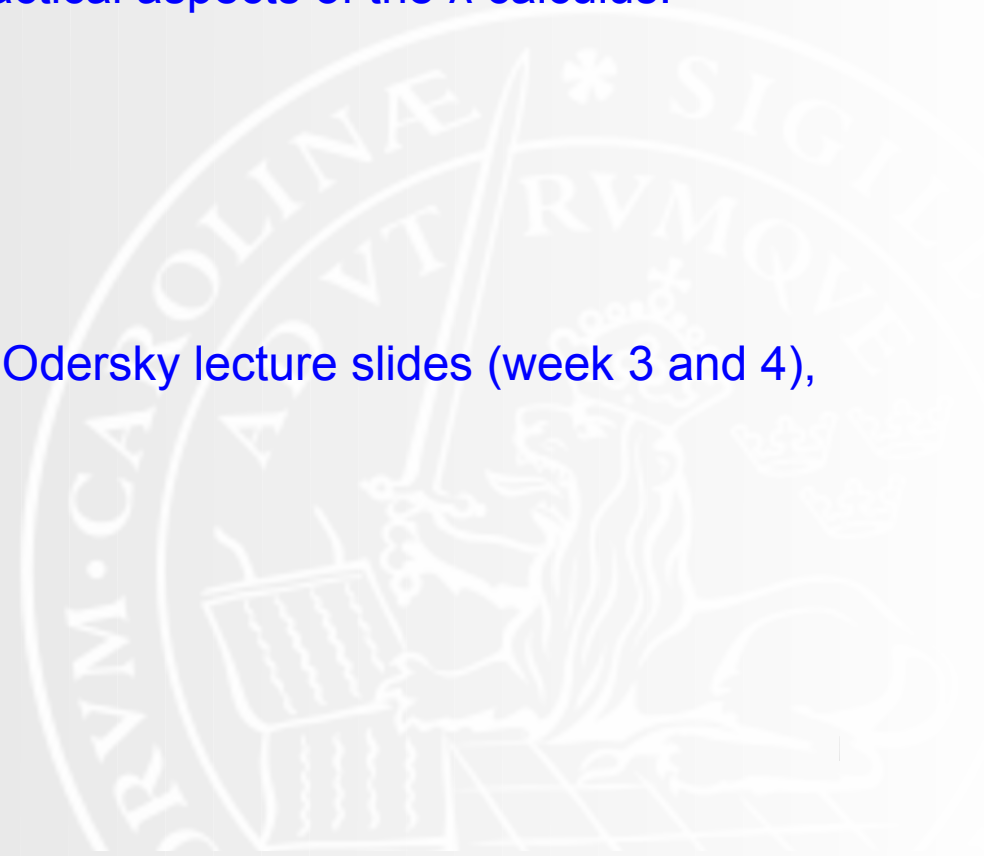


Note:

We omit a lot of the discussion of the practical aspects of the λ calculus. Specifically, this includes...

- programming techniques
- recursion
- behavioral equivalence of terms

These topics are being discussed in the Odersky lecture slides (week 3 and 4), and also in section 5.2 of TAPL.



Induction on the λ Calculus

Two induction principles

Like before, we have two ways to prove that properties are true of the untyped lambda calculus.

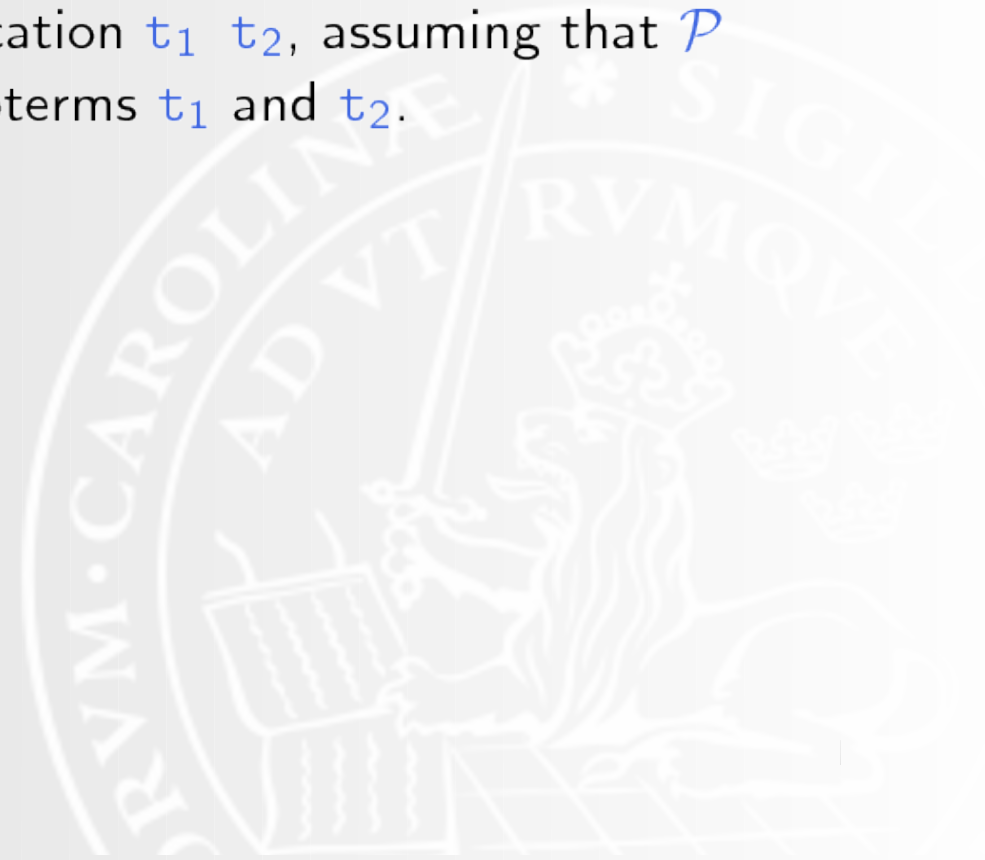
- Structural induction on terms
- Induction on a derivation of $\mathtt{t} \longrightarrow \mathtt{t}'$.

Let's look at an example of each.

Structural induction on terms

To show that a property $\mathcal{P}$ holds for all lambda-terms $\mathtt{t}$, it suffices to show that

- $\mathcal{P}$ holds when $\mathtt{t}$ is a variable;
- $\mathcal{P}$ holds when $\mathtt{t}$ is a lambda-abstraction $\lambda \mathtt{x}.\ \mathtt{t}_1$, assuming that $\mathcal{P}$ holds for the immediate subterm $\mathtt{t}_1$; and
- $\mathcal{P}$ holds when $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, assuming that $\mathcal{P}$ holds for the immediate subterms $\mathtt{t}_1$ and $\mathtt{t}_2$.



Structural induction on terms

To show that a property $\mathcal{P}$ holds for all lambda-terms $\mathtt{t}$, it suffices to show that

- $\mathcal{P}$ holds when $\mathtt{t}$ is a variable;
- $\mathcal{P}$ holds when $\mathtt{t}$ is a lambda-abstraction $\lambda\mathtt{x}.\ \mathtt{t}_1$, assuming that $\mathcal{P}$ holds for the immediate subterm $\mathtt{t}_1$; and
- $\mathcal{P}$ holds when $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, assuming that $\mathcal{P}$ holds for the immediate subterms $\mathtt{t}_1$ and $\mathtt{t}_2$.

N.b.: The variant of this principle where "immediate subterm" is replaced by "arbitrary subterm" is also valid. (Cf. *ordinary induction* vs. *complete induction* on the natural numbers.)

An example of structural induction on terms

Define the set of *free variables* in a lambda-term as follows:

$$FV(\mathtt{x}) = \{\mathtt{x}\}$$
$$FV(\lambda \mathtt{x}.\mathtt{t}_1) = FV(\mathtt{t}_1) \setminus \{\mathtt{x}\}$$
$$FV(\mathtt{t}_1\ \ \mathtt{t}_2) = FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2)$$

Define the *size* of a lambda-term as follows:

$$size(\mathtt{x}) = 1$$
$$size(\lambda \mathtt{x}.\mathtt{t}_1) = size(\mathtt{t}_1) + 1$$
$$size(\mathtt{t}_1\ \ \mathtt{t}_2) = size(\mathtt{t}_1) + size(\mathtt{t}_2) + 1$$

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

An example of structural induction on terms

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

Proof: By induction on the structure of $\mathtt{t}$.

- If $\mathtt{t}$ is a variable, then $|FV(\mathtt{t})| = 1 = size(\mathtt{t})$.
- If $\mathtt{t}$ is an abstraction $\lambda \mathtt{x}.\ \mathtt{t}_1$, then

$$
\begin{array}{lll}
 & |FV(\mathtt{t})| & \\
= & |FV(\mathtt{t}_1) \setminus \{\mathtt{x}\}| & \text{by defn} \\
\leq & |FV(\mathtt{t}_1)| & \text{by arithmetic} \\
\leq & size(\mathtt{t}_1) & \text{by induction hypothesis} \\
< & size(\mathtt{t}_1) + 1 & \text{by arithmetic} \\
= & size(\mathtt{t}) & \text{by defn.}
\end{array}
$$

An example of structural induction on terms

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

Proof: By induction on the structure of $\mathtt{t}$.

- If $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, then

$$
\begin{array}{lll}
 & |FV(\mathtt{t})| & \\
= & |FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2)| & \text{by defn} \\
\leq & |FV(\mathtt{t}_1)| + |FV(\mathtt{t}_2)| & \text{by arithmetic} \\
\leq & size(\mathtt{t}_1) + size(\mathtt{t}_2) & \text{by IH and arithmetic} \\
< & size(\mathtt{t}_1) + size(\mathtt{t}_2) + 1 & \text{by arithmetic} \\
= & size(\mathtt{t}) & \text{by defn.}
\end{array}
$$

Induction on derivations

Recall that the reduction relation is defined as the smallest binary relation on terms satisfying the following rules:

$$(\lambda x.t_1)\ v_2 \longrightarrow [x \mapsto v_2]t_1 \qquad \text{(E-AppAbs)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \qquad \text{(E-App2)}$$

Induction on derivations

Induction principle for the small-step evaluation relation.

To show that a property $\mathcal{P}$ holds for all derivations of $\mathtt{t} \longrightarrow \mathtt{t}'$, it suffices to show that

- $\mathcal{P}$ holds for all derivations that use the rule E-AppAbs;
- $\mathcal{P}$ holds for all derivations that end with a use of E-App1 assuming that $\mathcal{P}$ holds for all subderivations; and
- $\mathcal{P}$ holds for all derivations that end with a use of E-App2 assuming that $\mathcal{P}$ holds for all subderivations.

An example of induction on derivations

Theorem: if $t \longrightarrow t'$ then $FV(t) \supseteq FV(t')$.

We must prove, for all derivations of $t \longrightarrow t'$, that $FV(t) \supseteq FV(t')$.

An example of induction on derivations

Theorem: if $\mathtt{t} \longrightarrow \mathtt{t}'$ then $FV(\mathtt{t}) \supseteq FV(\mathtt{t}')$.

Proof: by induction on the derivation of $\mathtt{t} \longrightarrow \mathtt{t}'$. There are three cases:

An example of induction on derivations

Theorem: if $t \longrightarrow t'$ then $FV(t) \supseteq FV(t')$.

Proof: by induction on the derivation of $t \longrightarrow t'$. There are three cases:

- If the derivation of $t \longrightarrow t'$ is just a use of E-AppAbs, then t is $(\lambda x.t_1)v$ and t' is $[x \mapsto v]t_1$. Reason as follows:

$$\begin{aligned} FV(t) &= FV((\lambda x.t_1)v) \\ &= FV(t_1) \setminus \{x\} \cup FV(v) \\ &\supseteq FV([x \mapsto v]t_1) \\ &= FV(t') \end{aligned}$$

An example of induction on derivations

Theorem: if $t \longrightarrow t'$ then $FV(t) \supseteq FV(t')$.

Proof: by induction on the derivation of $t \longrightarrow t'$. There are three cases:

- If the derivation ends with a use of E-App1, then t has the form $t_1\ t_2$ and t' has the form $t_1'\ t_2$, and we have a subderivation of $t_1 \longrightarrow t_1'$

  By the induction hypothesis, $FV(t_1) \supseteq FV(t_1')$. Now calculate:

$$\begin{aligned} FV(t) &= FV(t_1\ t_2) \\ &= FV(t_1) \cup FV(t_2) \\ &\supseteq FV(t_1') \cup FV(t_2) \\ &= FV(t_1'\ t_2) \\ &= FV(t') \end{aligned}$$

- E-App2 is treated similarly.