Type Systems Course

Preliminaries (Chapter 2)

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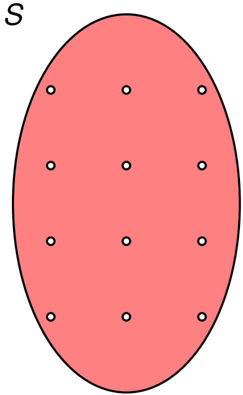
Sets, Relations, and Functions



Notation: $\{\ldots\}$

By comprehension: $\{x \in S | \ldots\}$

Cardinality / Size: $|S| = 12$, $|\mathbb{N}| = \aleph_0$

The empty set: $\emptyset$

Power set: $\mathcal{P}(T) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$
for $T = \{x, y\}$

A set S is *countable* if there is a one-to-one correspondence to $\mathbb{N}$

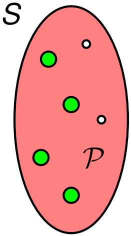
Sets, Relations, and Functions

Cartesian product: $X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$

n-place relation: $\mathcal{R} \subseteq \{S_1 \times S_2 \times \ldots \times S_n\}$
elements $s_1 \in S_1$ to $s_n \in S_n$ are related by $\mathcal{R}$ if $(s_1, \ldots, s_n) \in \mathcal{R}$

Sets, Relations, and Functions

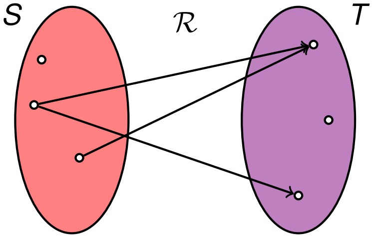
One-place relation / Predicate: $\mathcal{P}(S)$, $\mathcal{P} \subseteq S$



Sets, Relations, and Functions

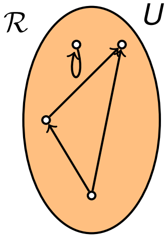
Two-place / Binary relation: $\mathcal{S}\mathcal{R}T$
$(s, t) \in \mathcal{R}$, for $s \in S$, $t \in T$

if $T = U$ and $S = U$, then $\mathcal{R}$ is a binary relation on U



domain,
domain($\mathcal{R}$)

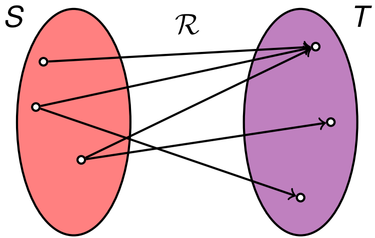
co-domain,
range,
range($\mathcal{R}$),



Sets, Relations, and Functions

Partial function: $\mathcal{R}$ is a partial function if whenever $(s, t_1) \in \mathcal{R}$ and $(s, t_2) \in \mathcal{R}$ we have $t_1 = t_2$

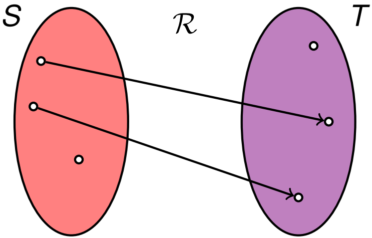
Is this a partial function?



Sets, Relations, and Functions

Partial function: $\mathcal{R}$ is a partial function if whenever $(s, t_1) \in \mathcal{R}$ and $(s, t_2) \in \mathcal{R}$ we have $t_1 = t_2$

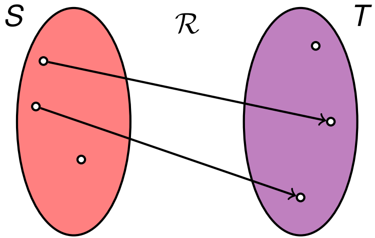
What about this one?



Sets, Relations, and Functions

Total function: a partial function $\mathcal{R}$ on *S* and *T* is a total function if $domain(\mathcal{R}) = S$

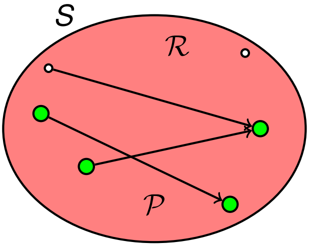
Is this a total function?



Sets, Relations, and Functions

Defined: the partial function $\mathcal{R}$ is *defined* on the argument s if $s \in domain(\mathcal{R})$, also written as $\mathcal{R}(s)\downarrow$

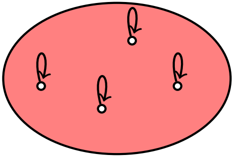
Preserved: if $\mathcal{R}$ is a relation on a set S with a predicate $\mathcal{P}$, then $\mathcal{P}$ is *preserved by* $\mathcal{R}$ if whenever we have $s_1 \mathcal{R} s_2$ and $\mathcal{P}(s_1)$ we also have $\mathcal{P}(s_2)$



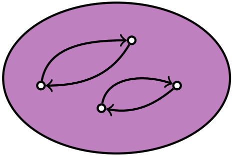
Ordered Sets

A binary relation $\mathcal{R}$ on S is ...

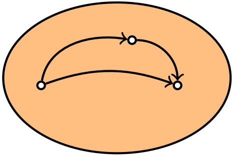
... **reflexive** if $\forall s \in S$
we have $(s, s) \in \mathcal{R}$



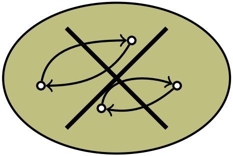
... **symmetric** if $\forall s, t \in S$
$(s, t) \in \mathcal{R} \implies (t, s) \in \mathcal{R}$



... **transitive** if $\forall s, t, u \in S$
$(s, t), (t, u) \in \mathcal{R} \implies (s, u) \in \mathcal{R}$



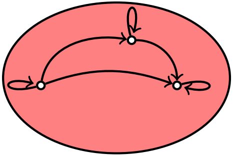
... **antisymmetric** if $\forall s, t \in S$
$(s, t), (t, s) \in \mathcal{R} \implies t = s$



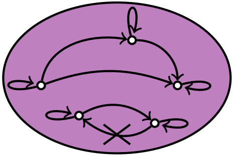
Ordered Sets

A relation $\mathcal{R}$ on a set S is called a ...

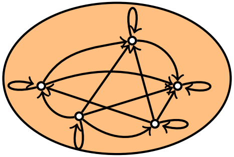
... **preorder** if it is *reflexive* and *transitive*, often denoted with $\leq$ or $\sqsubseteq$



... **partial order** if it is a *preorder* and *antisymmetric*



... **total order** if it is a *partial order* and for each pair $s, t \in S$ we have $s \leq t$ or $t \leq s$

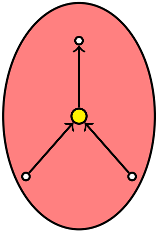


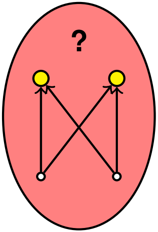
Ordered Sets

For a partial order $\leq$ on a set S ...

... an element $j \in S$ is a **join** or **least upper bound** if

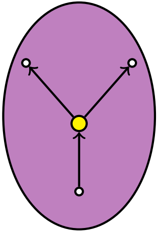
1. $s \leq j$ and $t \leq j$
2. for any $k \in S$, where $s \leq k$ and $t \leq k$, we have $j \leq k$





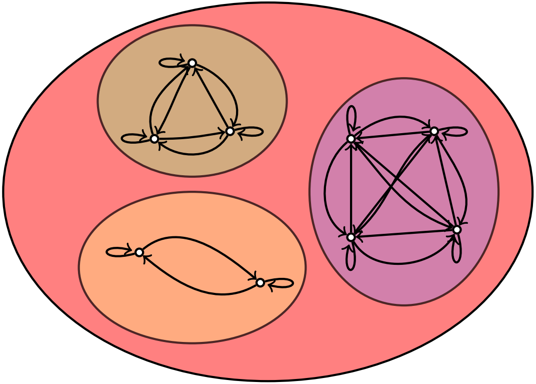
... an element $m \in S$ is a **meet** or **greatest lower bound** if

1. $m \leq s$ and $m \leq t$
2. for any element $n \in S$, where $n \leq s$ and $n \leq t$, we have $m \leq n$



Ordered Sets

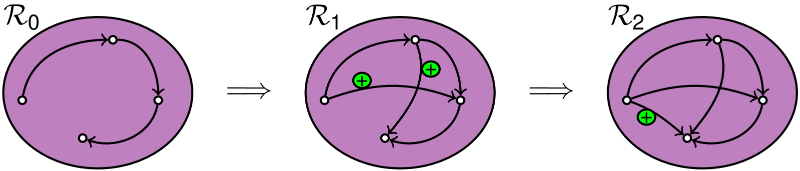
A relation $\mathcal{R}$ on a set S which is *reflexive*, *transitive*, and *symmetric* is called an **equivalence** on S (denoted with $\sim$)



Ordered Sets

The **reflexive closure** of a relation $\mathcal{R}$ on a set S is the *smallest reflexive* relation $\mathcal{R}'$ that contains $\mathcal{R}$, $\mathcal{R}' = \mathcal{R} \cup \{(s, s) | s \in S\}$

The **transitive closure** of a relation $\mathcal{R}$ on a set S is the *smallest transitive* relation $\mathcal{R}'$ that contains $\mathcal{R}$, denoted with
$\mathcal{R}^+ = \bigcup_{i \in \mathbb{N}} \mathcal{R}_i$ where
$\mathcal{R}_0 = \mathcal{R}$
$\mathcal{R}_{i+1} = \mathcal{R}_i \cup \{(s, u) | \text{for some } t, (s, t) \in \mathcal{R}_i \text{ and } (t, u) \in \mathcal{R}_i\}$



The **reflexive and transitive closure** is denoted $\mathcal{R}^*$

Ordered Sets

A **decreasing chain** on a preorder $\leq$ is a sequence of $s_1, s_2, s_3, ...$ of elements in S such that $s_{i+1} < s_i$ (these may be both finite and inifinite)

A preorder $\leq$ on a set S is called **well-founded** if it contains no infinite decreasing chains, e.g., the preorder $\leq$ on $\{..., -2, -1, 0, 1, ...\}$ is not well-founded

Sequences

A **sequence** is an ordered list of elements, e.g., a, b, c. Order matters, i.e., b, a, c is not the same sequence

A sequence containing the same elements of another sequence, but in a different order, is called a **permutation**

The **length** of a sequence $a = 2, 5, 7$ is given by $|a| = 3$

Sequences may be **concat**inated, e.g., $0, a$ gives us $0, 2, 5, 7$