



## Logic: A Summary

Jacek Malec

Dept. of Computer Science, Lund University, Sweden

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## Formal languages and syntax:

propositional variables:  $P, Q, R, S$

operators (connectives):  $\neg, \vee, \wedge$

formulae:  $P, \neg Q \wedge R, \neg(Q \vee R)$

*Language:*

the set of all well-formed formulae (wff):

$$\{P, Q, \neg P, \neg Q, P \wedge Q, P \vee Q, \dots\}$$



## Assigning truth values to symbols:

$P$  is TRUE

$Q$  is FALSE

*Interpretation:* an assignment to *all* of the variables.

It determines the truth values for more complex formulae:

$$\neg P \vee Q$$

$$\neg P \vee P$$

a tautology

$$\neg P \wedge P$$

a contradiction



## Logical equivalence:

$$Q \vee \neg P$$

$$\neg Q \vee P$$

$$\neg P \vee P$$

$$\neg P \wedge P$$

$$P \vee Q$$

$$\neg(\neg P \wedge \neg Q)$$

$$\neg P \vee Q$$

$$P \rightarrow Q$$



## Formal systems:

- Axioms
- Axiom schemas
- Rules of inference



## Rules of inference:

Modus Ponens:

$$\frac{A \quad A \rightarrow B}{B}$$

Conjunction:

$$\frac{A \quad B}{A \wedge B}$$



## Theoremhood:

- 1  $P \rightarrow Q$   
assume this is given as true
- 2  $Q \rightarrow R$   
assume this is given as true
- 3  $P$   
assume this is given as true
- 4  $Q$   
Modus Ponens using 1 and 3
- 5  $R$   
Modus Ponens using 2 and 4

Lines 1–4 constitute a *proof* of  $Q$ .  
Lines 1–5 constitute a proof of  $R$ .  
 $Q$  is a *theorem*.



## Satisfiability:

Is there an assignment to the variables such that the following formula is true?

$$\neg P \wedge (Q \vee \neg(R \wedge \dots))$$

Satisfiability problem is  $O(2^n)$

Similar questions:

- Is it a tautology?
- Is it a contradiction?



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## Expert or Rule-Based Systems:

```
(if (and p1 p2 ... pn) q)
```

### Tasks:

- prediction

```
(if (and john_is_in_the_building
        (not john_is_in_his_office)
        (not john_is_in_the_copy_room))
    john_is_in_the_conference_room)
```

- diagnosis

```
(if
  (and engine_is_running_hot
        engine_coolant_levels_within_spec)
  evidence_of_a_lubrication_problem)
```



## A note on Resolution:

It is a generalization of Modus Ponens

$$\frac{A_1 \vee A_2 \vee \dots \vee \neg C \vee \dots \vee A_m \quad B_1 \vee B_2 \vee \dots \vee C \vee \dots \vee B_n}{A_1 \vee A_2 \vee \dots \vee A_m \vee B_1 \vee B_2 \vee \dots \vee B_n}$$

Modus Ponens:

$$\frac{\neg P \vee Q \quad P}{Q}$$



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*Jacek, 59, Stockholm, Lund, Sweden, Pierre, table59, c, d, ...*
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- **Terms**: constants, variables, functions thereof
- **Atomic sentences**: relation over appropriate amount of terms  
*AgeOf(Jacek, 59), Bald(Jacek), 8 < x, YoungerThan(Jacek, fatherOf(Jacek)), P(x, y, z) locationOf(TJR048) = PDammgården, ...*



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- **Well-formed formulae**: as before plus  
 $\forall xA$  and  $\exists xA$  are wffs if  $A$  is a wff



## Quantifiers:

$$\forall x(\text{swedish} - \text{citizen}(x) \rightarrow \text{has} - \text{pnr}(x))$$

$$\exists y(\text{polish} - \text{citizen}(y) \wedge \text{has} - \text{pnr}(y))$$

$\forall xA$  and  $\exists xA$  are wffs if  $A$  is a wff

- scope of a quantifier
- free variable
- closed formula
- ground formula



## Formal System for FOPC:

language of FOPC, axioms + RES and UI

where Universal Instantiation:

$$\frac{\forall xA}{A'(x \rightarrow t)}$$

e.g. from

$$\forall x, y(Pit(x, y) \rightarrow Breeze(x, y + 1) \wedge Breeze(x + 1, y))$$

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$$Pit(1, 2) \rightarrow Breeze(1, 3) \wedge Breeze(2, 2),$$

and

$$Pit(2, 1) \rightarrow Breeze(2, 2) \wedge Breeze(3, 1),$$

and ...



## Theories:

$$\forall x, y \neg clown(x) \vee loves(y, x)$$

Everybody loves a clown.

$$\forall x, y \neg winner(x) \vee \neg game(y) \vee \neg plays(x, y) \vee wins(x, y)$$

A winner wins every game (s)he plays.

Pattern:

$$\forall x_1, \dots, x_n A$$

where  $A$  is in CNF



## Logically equivalent formulae:

1.

$$\forall x, y (clown(x) \rightarrow loves(y, x))$$

$$\forall x (clown(x) \rightarrow \forall y (loves(y, x)))$$

2.

$$\forall x A \leftrightarrow \neg \exists x \neg A$$

$$\exists x A \leftrightarrow \neg \forall x \neg A$$

Example:

$$(\forall x, y) \neg clown(x) \vee loves(y, x)$$

$$(\forall y) \neg ((\exists x) (clown(x) \wedge \neg loves(y, x)))$$

$$(\forall x) clown(x) \rightarrow \neg ((\exists y) \neg loves(y, x))$$



## Theorem proving:

Show  $\text{loves}(Pia, Kalle)$  given axioms:

- 1  $\forall x, y \text{clown}(x) \rightarrow \text{loves}(y, x)$
- 2  $\text{clown}(Kalle)$

Proof:

- 1  $\forall x, y \text{clown}(x) \rightarrow \text{loves}(y, x)$  (AXIOM)
- 2  $\text{clown}(Kalle)$  (AXIOM)
- 3  $\forall y \text{clown}(Kalle) \rightarrow \text{loves}(y, Kalle)$   
UI  $x \rightarrow Kalle$
- 4  $\text{clown}(Kalle) \rightarrow \text{loves}(Pia, Kalle)$   
UI  $y \rightarrow Pia$
- 5  $\text{loves}(Pia, Kalle)$   
MP 2,4



## Search, search everywhere...

Theorem proving

is

a search in the space of proofs