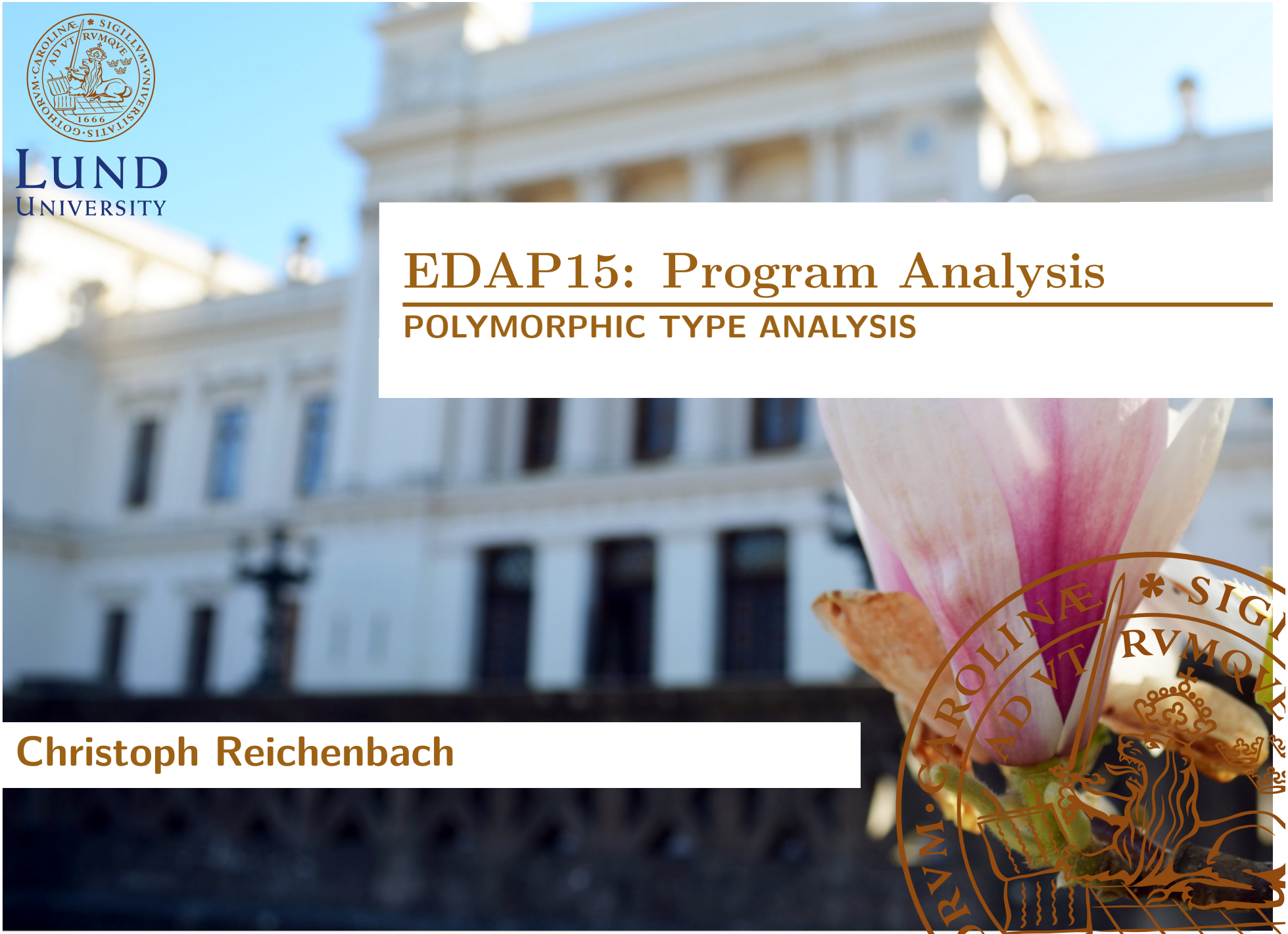

EDAP15: Program Analysis

POLYMORPHIC TYPE ANALYSIS

Christoph Reichenbach

Announcements

- Exercises start on Friday
- Call for Student Representative
- Video for second lecture damaged / unusable
  - Will record again later this semester
- Online office hours "on demand", please e-mail me

The Language LINGA

$$
\begin{array}{lcl}
\mathit{expr} & ::= & \langle \mathit{val} \rangle \\
 & | & \mathit{id} \\
 & | & \text{let } \mathit{id} = \langle \mathit{expr} \rangle \text{ in } \langle \mathit{expr} \rangle \\
 & | & \text{nil} \\
 & | & \text{cons } (\langle \mathit{expr} \rangle, \langle \mathit{expr} \rangle) \\
 & | & \langle \mathit{expr} \rangle \text{ plus } \langle \mathit{expr} \rangle \\
 & | & \langle \mathit{expr} \rangle >= \langle \mathit{expr} \rangle \\
 & | & \text{if } \langle \mathit{expr} \rangle \text{ then } \langle \mathit{expr} \rangle \text{ else } \langle \mathit{expr} \rangle
\end{array}
$$

CONS(1, CONS(2, NIL))

$$
\begin{array}{lcl}
\mathit{val} & ::= & \mathit{nat} \\
 & | & \text{true} \mid \text{false}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{ty} & ::= & \textsc{Int} \\
 & | & \textsc{Bool} \\
 & | & \textsc{List}\ [\langle \mathit{ty} \rangle] \\
 & | & \mathit{tyvar}
\end{array}
$$

$$
\mathit{tyvar} \quad ::= \quad \underline{\alpha \mid \beta \mid \gamma} \mid \ \ldots
$$

Parametric Types and Principal Types

nil : τ

- Problem last week:
  - $\mathtt{p} : \tau$ may have infinitely many τ, can't process all
  - One instance of more general problem:
    Having too many such τ makes analysis inefficient
- General approach: Find **Principal Type**
  - *Single* type that summarises all other types
- Here: use **Parametric Types** with **Type Variables**:
  - $\textsc{List}[\alpha]$ summarises $\textsc{List}[\textsc{Int}]$, $\textsc{List}[\textsc{Bool}]$, $\textsc{List}[\textsc{List}[\ldots]]$

Parametric Type

Typing Rules for Parametric Types

$$\frac{}{\mathsf{true} : \textsc{Bool}}\ (\textit{t-true}) \qquad \frac{}{\mathsf{false} : \textsc{Bool}}\ (\textit{t-false}) \qquad \frac{v \in \textit{nat}}{v : \textsc{Int}}\ (\textit{t-nat})$$

$$\frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 \ \mathsf{plus}\ e_2 : \textsc{Int}}\ (\textit{t-plus}) \qquad \frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 >= e_2 : \textsc{Bool}}\ (\textit{t-ge}) \qquad \frac{\underline{x}.\mathtt{ty} = \tau}{\underline{x} : \tau}\ (\textit{t-var})$$

$$\frac{e_1 : \textsc{Bool} \quad e_2 : \tau \quad e_3 : \tau}{\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 : \tau}\ (\textit{t-if}) \qquad \frac{e_1 : \tau_1 \quad \underline{x}.\mathtt{ty} = \tau_1 \quad e_2 : \tau_2}{\mathsf{let}\ \underline{x} = e_1\ \mathsf{in}\ e_2 : \tau_2}\ (\textit{t-let})$$

$$\frac{}{\mathsf{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\mathsf{cons}(e_1, e_2) : \textsc{List}[\tau]}\ (\textit{t-cons})$$

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$$\frac{}{\text{nil} : \textsc{List}[\alpha]}\ (t\text{-}nil) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\text{cons}(e_1, e_2) : \textsc{List}[\tau]}\ (t\text{-}cons)$$

$x.ty = LIST[\alpha] = LIST[INT]$

$\alpha = INT$

$$\frac{\dfrac{}{nil : \tau_1 = LIST[\alpha]}\ t\text{-}nil \quad \underline{x}.ty = \tau_1 = LIST[\alpha] \quad \dfrac{\dfrac{1 \in nat}{1 : INT}\ t\text{-}nat \quad \dfrac{\underline{x}.ty = LIST[INT]}{\underline{x} : LIST[INT]}\ t\text{-}var}{cons(1, \underline{x}) : \tau_2 = LIST[INT]}\ t\text{-}cons}{\text{let } \underline{x} = \text{nil in cons}(1, \underline{x}) : \textsc{List}[\textsc{Int}]}\ (t\text{-}let)$$

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Originally $\underline{x}.\texttt{ty} = \textsc{List}[\alpha]$
Must merge $\textsc{List}[\alpha] = \textsc{List}[\textsc{Int}]$
Analogous to variable types

$\underline{x}.\texttt{ty} = \cancel{\textsc{List}[\alpha]}\ \textsc{List}[\textsc{Int}]$

$\boxed{\alpha.\texttt{ty} = \textsc{Int}}$

$$\frac{\dfrac{}{\text{nil} : \textsc{List}[\alpha]}\ (t\text{-}nil) \qquad \underline{x}.\texttt{ty} = \textsc{List}[\alpha] \qquad \dfrac{\dfrac{1 \in nat}{1 : \textsc{Int}}\ (t\text{-}nat) \qquad \dfrac{\underline{x}.\texttt{ty} = \textsc{List}[\textsc{Int}]}{\underline{x} : \textsc{List}[\textsc{Int}]}}{\text{cons}(1, \underline{x}) : \textsc{List}[\textsc{Int}]}}{\text{let } \underline{x} = \text{nil in cons}(1, \underline{x}) : \textsc{List}[\textsc{Int}]}\ (t\text{-}let)$$

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$\tau = \textsc{List}[\alpha]$

$\textsc{List}[\tau] = \textsc{List}[\alpha]$

$\tau = \textsc{List}[\tau]$

$$\frac{\dfrac{}{\text{nil} : \tau = \textsc{List}[\alpha]}\ t\text{-}nil \qquad \dfrac{}{\text{nil} : \textsc{List}[\tau] = \textsc{List}[\alpha]}\ t\text{-}nil}{\text{cons}(\text{nil}, \text{nil}) :}\ (t\text{-}cons)$$

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Circular type — *t-cons* requires:

$$\begin{aligned} \tau &= \textsc{List}[\alpha] \\ \textsc{List}[\tau] &= \textsc{List}[\alpha] \end{aligned}$$

$$\boxed{\alpha.\texttt{ty} = \textsc{List}[\alpha]}$$

$$\frac{\dfrac{}{\text{nil} : \textsc{List}[\alpha]}\ (t\text{-}nil) \qquad \dfrac{}{\text{nil} : \textsc{List}[\alpha]}\ (t\text{-}nil)}{\text{cons}(\text{nil}, \text{nil}) : \ ?}\ (t\text{-}cons)$$

Type Variable Freshness

- Our typing rule for nil doesn't work as intended:
  All nil use the same α in their type
  $\Longrightarrow$ all lists must have the same type

$$\frac{}{\mathsf{nil} : \textsc{List}[\alpha]}\ (\textit{t-nil}) \qquad \frac{e_1 : \tau \quad e_2 : \textsc{List}[\tau]}{\mathsf{cons}(e_1,\ e_2) : \textsc{List}[\tau]}\ (\textit{t-cons})$$

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- Fix: We create a *fresh* type variable for every nil

$\alpha_1 =$

$\alpha_2 = \textsc{List}[\alpha_1]$

$$\frac{\dfrac{\alpha_1 \text{ fresh}}{\text{nil} : \textsc{List}[\alpha_1]}\ t\text{-nil} \qquad \dfrac{\alpha_2 \text{ fresh}}{\text{nil} : \textsc{List}[\alpha_2]}\ t\text{-nil}}{\text{cons}(\text{nil}, \text{nil}) : \textsc{List}[\textsc{List}[\alpha_1]]} \quad \textsc{List}[\alpha_2] = \textsc{List}[\textsc{List}[\alpha_1]]$$

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Parametric Types in Practice

- Widely used today, e.g. *Generics* in Java:

Java	**Scala**
`List<E>`	`List[A]`
`Set<E>`	`Set[A]`
`Map<K, V>`	`Map[K, V]`

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- Scala and others also support parametric *tuple types*:

Scala	**Ocaml/SML**	**Common**
`(A, B, C)`	`'a * 'b * 'c`	$\alpha \times \beta \times \gamma$

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`(A, B, C)`	`'a * 'b * 'c`	$\alpha \times \beta \times \gamma$

- We often combine tuple and function types when inferring types of functions:

$$\texttt{countOccurrencesInList} : \underline{\textsc{List}[\alpha]} \times \underline{\alpha} \to \underline{\textsc{Int}}$$

More Uses for Type Variables

- Type variables help us defer decisions about types when we have no information
- Recall:

$$\frac{\underline{x}.\texttt{ty} = \tau}{\underline{x} : \tau}\ (\textit{t-var})$$

- This rule won't help us type e.g. function parameters:

Python

```
def f(x) :
    return (x, x)
```

$f : \alpha \to \alpha \times \alpha$

- Can't apply *t-var* if we have never seen $\underline{x}$ before

More Uses for Type Variables

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Python

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- Can't apply *t-var* if we have never seen $\underline{x}$ before
- Instead, we can use a different rule for variables:

$$\frac{\underline{x}.\texttt{ty} = \alpha \quad \alpha\ \textit{fresh}}{\underline{x} : \alpha}\ (\textit{t-var}')$$

Summary

- We often need recursive types in our analyses
- As a result, some expressions may have an unbounded number of types
- We can usually use **type variables** to present these types practically
- This produces **principal types** if we can summarise *all* types
- **Parametric types** (or *parametrically polymorphic*) types arise frequently
- Correctly using expressions with type variables may require us to produce **fresh type variables**
- Open question:
  How *do* we merge type variables in equations?

$$\underline{\textsc{List}[\alpha_1]} = \underline{\textsc{List}[\textsc{List}[\alpha_2]]}$$

Type Inference with Variables: Example

Python

```
def gen(a:map, b:set):
  m = {}
  for v in b:
      if v in a.keys():
          x = a[v]
          m[x] = x
  return m
```

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3      if v in a.keys():
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5          m[x] = x
6  return m
```

Extract *typings*:

$\mathtt{y} : \tau$

a	:	$\mathtt{map}[\beta_1, \beta_2]$
b	:	$\mathtt{set}[\gamma]$
gen	:	$(\mathtt{map}[\beta_1, \beta_2] \times \mathtt{set}[\gamma]) \rightarrow \xi$

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1 m	:	$\texttt{map}[\alpha_1,\ \alpha_2]$

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a	:	$\mathtt{map}[\beta_1,\ \beta_2]$
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gen	:	$(\mathtt{map}[\beta_1,\beta_2],\mathtt{set}[\gamma]) \rightarrow \xi$
m	:	$\mathtt{map}[\alpha_1,\ \alpha_2]$
v	:	γ

1

2

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```

Extract *typings*:

$y : \tau$

Extract *equality constraints*:

$\tau_1 = \tau_2$

	a	: map[β_1, β_2]
	b	: set[γ]
	gen	: (map[β_1,β_2],set[γ])$\rightarrow\xi$
1	m	: map[α_1, α_2]
2	v	: γ
3	v	: β_1
	γ	$=$ β_1

Type Inference with Variables: Example

Python

```
def gen(a:map, b:set):
1   m = {}
2   for v in b:
3       if v in a.keys():
4           x = a[v]
5           m[x] = x
6   return m
```

Extract *typings*:

$y : \tau$

Extract *equality constraints*:

$\tau_1 = \tau_2$

	a	:	map[β_1, β_2]
	b	:	set[γ]
	gen	:	(map[β_1,β_2],set[γ])$\rightarrow\xi$
1	m	:	map[α_1, α_2]
2	v	:	γ
3	v	:	β_1
	γ	$=$	β_1
4	x	:	α_3
	a	:	map[γ, α_3]
	map[β_1, β_2]	$=$	map[γ, α_3]

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	b	:	$\texttt{set}[\gamma]$
	gen	:	$(\texttt{map}[\beta_1,\beta_2],\texttt{set}[\gamma]) \rightarrow \xi$
1	m	:	$\texttt{map}[\alpha_1,\ \alpha_2]$
2	v	:	γ
3	v	:	β_1
	γ	$=$	β_1
4	x	:	α_3
	a	:	$\texttt{map}[\gamma,\ \alpha_3]$
	$\texttt{map}[\beta_1,\ \beta_2]$	$=$	$\texttt{map}[\gamma,\ \alpha_3]$
5	m	:	$\texttt{map}[\alpha_2,\ \beta_2]$
	$\texttt{map}[\alpha_1,\ \alpha_2]$	$=$	$\texttt{map}[\alpha_2,\ \beta_2]$

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1	m	:	$\texttt{map}[\alpha_1,\ \alpha_2]$
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3	v	:	β_1
	γ	$=$	β_1
4	x	:	α_3
	a	:	$\texttt{map}[\gamma,\ \alpha_3]$
	$\texttt{map}[\beta_1,\ \beta_2]$	$=$	$\texttt{map}[\gamma,\ \alpha_3]$
5	m	:	$\texttt{map}[\alpha_2,\ \beta_2]$
	$\texttt{map}[\alpha_1,\ \alpha_2]$	$=$	$\texttt{map}[\alpha_2,\ \beta_2]$
6	m	:	ξ
	ξ	$=$	$\texttt{map}[\alpha_1,\ \alpha_2]$

How do we solve this automatically?

Type Inference: Constraints

Typings:

$$
\begin{array}{rcl}
\mathsf{a} & : & \texttt{map}[\beta_1,\ \beta_2] \\
\mathsf{b} & : & \texttt{set}[\gamma] \\
\mathsf{gen} & : & (\texttt{map}[\beta_1,\beta_2],\texttt{set}[\gamma]) \rightarrow \xi \\
\mathsf{m} & : & \texttt{map}[\alpha_1,\ \alpha_2] \\
\mathsf{v} & : & \gamma \\
\mathsf{v} & : & \beta_1 \\
\mathsf{x} & : & \alpha_3 \\
\mathsf{a} & : & \texttt{map}[\gamma,\ \alpha_3] \\
\mathsf{m} & : & \texttt{map}[\alpha_2,\ \beta_2] \\
\mathsf{m} & : & \xi
\end{array}
$$

Type Equality Constraints:

$$
\begin{array}{rcl}
\gamma & = & \beta_1 \\
\texttt{map}[\beta_1,\ \beta_2] & = & \texttt{map}[\gamma,\ \alpha_3] \\
\texttt{map}[\alpha_1,\ \alpha_2] & = & \texttt{map}[\alpha_2,\ \beta_2] \\
\xi & = & \texttt{map}[\alpha_1,\ \alpha_2]
\end{array}
$$

Unification

$$
\begin{aligned}
\gamma &= \beta_1 \\
\texttt{map}[\beta_1,\ \beta_2] &= \texttt{map}[\gamma,\ \alpha_3] \\
\texttt{map}[\alpha_1,\ \alpha_2] &= \texttt{map}[\alpha_2,\ \beta_2] \\
\xi &= \texttt{map}[\alpha_1,\ \alpha_2]
\end{aligned}
$$

- *Unification* describes the problem of solving such equations
- Some unification problems are undecidable
  - *Subtyping* in particular usually leads to undecidability
- Our problem has an efficient (near-linear) solution:
  - Given a *worklist* of equality constraints:
  - Remove and process one constraint at a time
  - If constraint has form $\alpha = T$: replace $\alpha \to T$
  - Otherwise, break equation into smaller equalities, add to worklist
  - . . . plus some minor tweaks

Unification

$$
\begin{array}{rcl}
\gamma & = & \beta_1 \\
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\xi & = & \texttt{map}[\alpha_1,\ \alpha_2]
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First, let us simplify our representation

Type Constructors

- Recall Parametric Types:
  - `Set[`α`]`
  - `Map[`α`, `β`]`
- Type constructors: things like `Set`, `Map`
  - Take type parameters α, β
  - Build new type

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  - $\rightarrow$: constructs function types

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- General notation: $C_i^k(\tau_1, \ldots, \tau_k)$
  - E.g.: $\texttt{int} \rightarrow \texttt{string} = C_{\rightarrow}^2(\texttt{int}, \texttt{string})$
  - E.g.: $Set[Set[\texttt{int}]] = C_{\texttt{Set}}^1(C_{\texttt{Set}}^1(\texttt{int}))$
- k: arity of type constructor
- i: globally unique identifier for constructor

Type Unification

- Each equation has one of these forms:

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$

(Martelli and Montanari, 1982, based on Robinson, 1965)

Type Unification

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   - Otherwise: Replace by equations:

$$\begin{array}{lcl} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

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set $\{\ldots\}$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Solution: Replace $C_i^k(\tau_1, \ldots, \tau_k)$ for α everywhere

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   ...but what about $\alpha = \textsc{List}[\alpha]$?

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4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Solution: Replace $C_i^k(\tau_1, \ldots, \tau_k)$ for α everywhere
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots) \Rightarrow \boxed{\textit{Type Error}}$ *($\leftarrow$ Occurs Check)*

(Martelli and Montanari, 1982, based on Robinson, 1965)

Example (Continued)

1. $\alpha = \alpha$ (trivial)
2. $\alpha = \beta$
   - Replace β with α
3. $C_i^k(\tau_1^a, \ldots, \tau_k^a) = C_j^l(\tau_1^b, \ldots, \tau_l^b)$
   - $\boxed{\textit{Type Error}}$ if $i \neq j$ or $k \neq l$
   - Otherwise: Replace by:

$$\begin{array}{ccc} \tau_1^a & = & \tau_1^b \\ \ldots & & \ldots \\ \tau_k^a & = & \tau_k^b \end{array}$$

4. $\alpha = C_i^k(\tau_1, \ldots, \tau_k)$
   - Replace $C_i^k(\tau_1, \ldots, \tau_k)$ for α
   - **Except:** $\alpha = C_i^k(\ldots, \alpha, \ldots)$
     $\Rightarrow$ $\boxed{\textit{Type Error}}$

gen : map[β_1, β_2] × set[β_1] → ξ

map[α_1, α_2]

$\gamma = \beta_1$

map[β_1, β_2] = map[γ, α_3]

map[α_1, α_2] = map[α_2, β_2]

ξ = map[α_1, α_2]

$\beta_1 = \beta_1$

$\beta_2 = \alpha_3$

$\alpha_1 = \alpha_2$

$\alpha_1 = \beta_2$

α_3

gen: map[β_1, α_2] × set[β_1] → map[α_2, α_2]

Substituting "Everywhere"?

- The Martelli/Montanari algorithm asks us to "replace type variables everywhere":
  - "Solution: Replace β with α everywhere"
  - "Solution: Replace $C_i^k(\tau_1, \ldots, \tau_k)$ for α everywhere"
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    - Replace everywhere in worklist
    - Replace everywhere in solutions (e.g., symbol table)
  - **Update Lists**:
    - 'Substitute systematically', but on demand, storing pending updates
  - **Stateful type variables**: (*my recommendation*)
    - Type variables remember their bindings, e.g. in α`.ty`
    - Some challenges with nontrivial merges

Summary

- During type analysis, we often encounter nontrivial equations over types
- To check these and extract relevant equalities, we use **Unification**
- The **Martelli/Montanari algorithm** is efficient for the types we have discussed so far
- Input:
  - A list of equations over types
- Output:
  - Bindings to type variables
  - Type variables such as α may be:
    - Replaced by a concrete type, such as $\textsc{Int}$
    - Replaced by another type variable, such as β
    - Replaced by a partially abstract type, such as $\textsc{List}[\gamma]$

Merging Variables

- Consider solving:

$$\begin{array}{rcl} \alpha & = & \beta \\ \beta & = & \gamma \\ \gamma & = & \delta \\ \delta & = & \xi \end{array}$$

- Implementing unification with stateful variables naively can make it costly to figure out the "real" type of α:

$$\boxed{\alpha\,\texttt{.ty} = \beta}$$
$$\boxed{\beta\,\texttt{.ty} = \gamma}$$
$$\boxed{\gamma\,\texttt{.ty} = \delta}$$
$$\boxed{\delta\,\texttt{.ty} = \xi}$$

- Fast unification implementations instead use UNION-FIND datastructures

Union-Find Datastructures

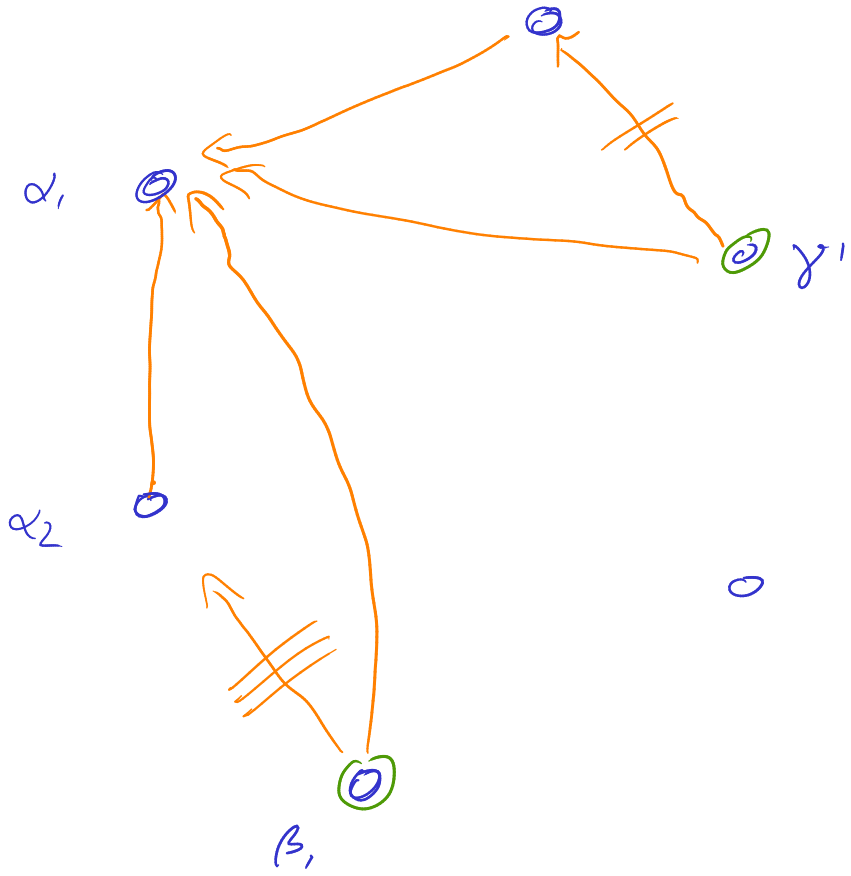
Java

```java
public class UFSet {
  UFSet repr = null;

  // Find & update representative
  public UFSet find() {
    UFSet r = this;
    while (r.repr != null) {
      r = r.repr;
    }
    this.repr = r;
    return r;
  }

  public void union(UFSet other) {
    other = other.find();
    UFSet r = this.find();
    // we can update r or other
    if (r != other) {
      other.repr = r;
  } }

  public boolean equals(UFSet o) {
    return this.find() == o.find();
} }
```



Summary

- UNION-FIND datastructure can speed up type variable merging
- Type variables represent a set of equivalent variables
- Each set has one representative
- *find* operation finds that representative
  - updates cached references to it
- *union*(v_1, v_2) operation finds representatives r_1, r_2 of two variables
  - If $r_1 \neq r_2$, v_1, v_2 in different set
  - Then, update either representative of v_1 to now be v_2, or vice-versa
  - High-performance implementations make this decision based on:
    - set size
    - estimated "depth" of representative chains ('*rank*')

Towards Real Languages: TEAL-0

```
module    ::= ⟨import⟩⋆ ⟨decl⟩⋆

import    ::= import ⟨qualified⟩ ;

qualified ::= id
            | ⟨qualified⟩ :: id

decl      ::= ⟨vardecl⟩ ;
            | fun id ( ⟨formals⟩? ) ⟨opttype⟩ = ⟨stmt⟩

vardecl   ::= var id ⟨opttype⟩
            | var id ⟨opttype⟩ := ⟨expr⟩;

formals   ::= id ⟨opttype⟩
            | id ⟨opttype⟩ , ⟨formal⟩

opttype   ::= : ⟨type⟩
            | ε

type      ::= int | string | any
            | array [ ⟨type⟩ ]

block     ::= { ⟨stmt⟩⋆ }
```

```
expr      ::= ⟨expr⟩ ⟨binop⟩ ⟨expr⟩
            | not ⟨expr⟩
            | ( ⟨expr⟩ ⟨opttype⟩ )
            | ⟨expr⟩ [ ⟨expr⟩ ]
            | id ( ⟨actuals⟩? )
            | [ ⟨actuals⟩? ]
            | new ⟨type⟩ ( ⟨expr⟩ )
            | int | string | null
            | id

actuals   ::= expr
            | expr, ⟨actuals⟩

binop     ::= + | - | * | / | %
            | == | != | < | <= | >= | >
            | or | and

stmt      ::= ⟨vardecl⟩
            | ⟨expr⟩ ;
            | ⟨expr⟩ := ⟨expr⟩ ;
            | ⟨block⟩
            | return ⟨expr⟩ ;
            | if ⟨expr⟩ ⟨block⟩ else ⟨block⟩
            | if ⟨expr⟩ ⟨block⟩
            | while ⟨expr⟩ ⟨block⟩
```

Unification, Types, and Re-use

int
String

Teal-0

```
fun id(x) = return x;

var b := id("foo")
```

α α

:String :String

id: $\alpha \to \alpha$
String $\to$ String

- What are the types here?

Unification, Types, and Re-use

Teal-0

```
fun id(x:α1):α2 = return x:α1;

var b:β1 := id("foo":string):β2
```

- What are the types here?
- Straightforward to solve with unification:

$$
\begin{array}{lcl}
\texttt{x} & : & \alpha_1 \\
\texttt{id} & : & \alpha_1 \rightarrow \alpha_2 \\
\texttt{b} & : & \beta_1 \\
\alpha_1 & = & \texttt{string} \\
\alpha_2 & = & \alpha_1 \\
\beta_2 & = & \alpha_2 \\
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\end{array}
$$

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\beta_1 & = & \texttt{string}
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$$

Unification, Types, and Re-use

Teal-0

```
fun id(x) = return x;

var b := id("foo")
var c := id(15)
```

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\texttt{x} & : & \texttt{string} \\
\texttt{id} & : & \texttt{string} \rightarrow \texttt{string} \\
\texttt{b} & : & \texttt{string} \\
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\alpha_1 & = & \texttt{string} \\
\alpha_2 & = & \alpha_1 \\
\beta_2 & = & \alpha_2 \\
\beta_1 & = & \beta_2
\end{array}
$$

- ... Now we also have $\alpha_1 = \texttt{int} \implies$ *type error*!

Unification, Types, and Re-use

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\beta_2 & = & \alpha_2 \\
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\end{array}
$$

- ... Now we also have $\alpha_1 = \texttt{int} \implies$ *type error*!

`id` function doesn't work for both `string` and `int`!

Type Schemes

- Recall:

$$\frac{}{\mathsf{nil} : \textsc{List}[\alpha]} \quad \Longrightarrow \quad \frac{\alpha \quad \textit{fresh}}{\mathsf{nil} : \textsc{List}[\alpha]}$$

```
          id: α → α
var b := id("foo");
```

```
          id: α → α
var b := id("foo");
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- Recall:

$$\frac{}{\text{nil} : \textsc{List}[\alpha]} \quad \Longrightarrow \quad \frac{\alpha \quad \textit{fresh}}{\text{nil} : \textsc{List}[\alpha]}$$

- Polymorphism with user-defined types:
  - *Type Schemes* (or *Polytypes*):
    (1) "normal" polymorphic type: $\alpha \to \alpha$
    (2) variables to replace by fresh ones: $\{ \alpha \}$

  - *Instantiate* type schemes with fresh type variables on demand

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        id: α → α
var b := id("foo");
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    short notation for (1)+(2): $\forall \alpha . \alpha \to \alpha$
  - `id` : $\forall \alpha . \alpha \to \alpha$
  - *Instantiate* type schemes with fresh type variables on demand:

id: $\alpha_2 \to \alpha_2 [\alpha \mapsto \alpha_2]$

```
var b := id("foo");
```

id: $\alpha_3 \to \alpha_3 [\alpha \mapsto \alpha_3]$

```
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```

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```
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```

```
        id: α3 → α3
var b := id("foo");
```

Using Type Schemes

- Instantiating type schemes:

$\beta_1 \ldots \beta_n$

$$\frac{x.\texttt{ty} = \forall \alpha_1, \ldots, \alpha_n.\tau \quad \beta_i \ \mathit{fresh}, i \in \{1, \ldots, n\}}{x : \tau[\alpha_1 \mapsto \beta_1, \ldots, \alpha_n \mapsto \beta_n]} \ (\textit{t-var-inst})$$

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How do we find $\mathcal{T}$?

Finding Type Schemes (1/3)

Teal-0

```
fun id(x) = return x;

var b = id("foo");
var c = id(17);
```

- When should we build the schema for `id`?

Finding Type Schemes (1/3)

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  - **After all unification**
  - **Before all unification**
  - **During Unification**

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    Too early:
    - Our first type constraint was: `id` : $\alpha_1 \to \alpha_2$
    - However, `id` : $\forall \alpha_1, \alpha_2 . \alpha_1 \to \alpha_2$ would be wrong:
      Tells us nothing about connection between α_1 and α_2
  - **During Unification**

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      Tells us nothing about connection between α_1 and α_2
  - **During Unification:**
    - Must abstract *after* unifying all variables that "matter" to `id`
    - Must abstract *before* we use `id`'s type

Finding Type Schemes (2/3)

Teal-0

```
fun f(x0, x1, x2) = return g(x0 - 1, x1, x2);

fun g(y0, y1, y2) =
    if y0 == y1 {
        return y1;
    } else {
        return f(y1, y0, y2);
    }
```

Finding Type Schemes (2/3)

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```

***Mutual dependency*: Can't fully analyse one before the other**

Finding Type Schemes (3/3)

- When functions call each other: must analyse them together
- Generalises to indirect calls
- Find *dependencies*:
  - if `f` calls `g`:
    - `f` *depends* (*directly*) on `g`
  - if `f` *depends* on `g` and `g` *depends* on `h`:
    - `f` *depends* on `h`
- `f` *depends* on `g`: Can't build schema for `f` before analysing `g`

$\Rightarrow$ Analyse such `f` and `g` together

Polymorphic Type Inference Procedure

f_0

f_1 f_2

f_4 f_3

f_5

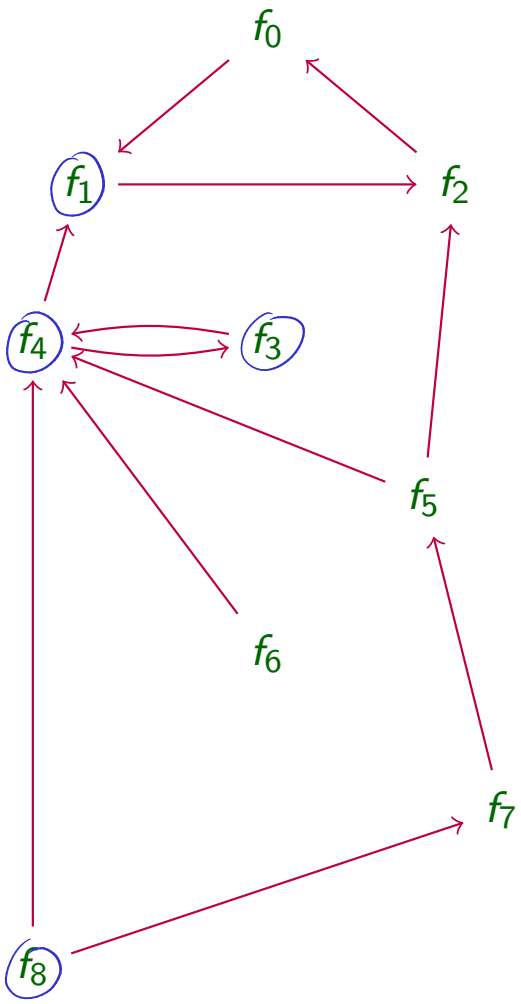
f_6

f_7

f_8

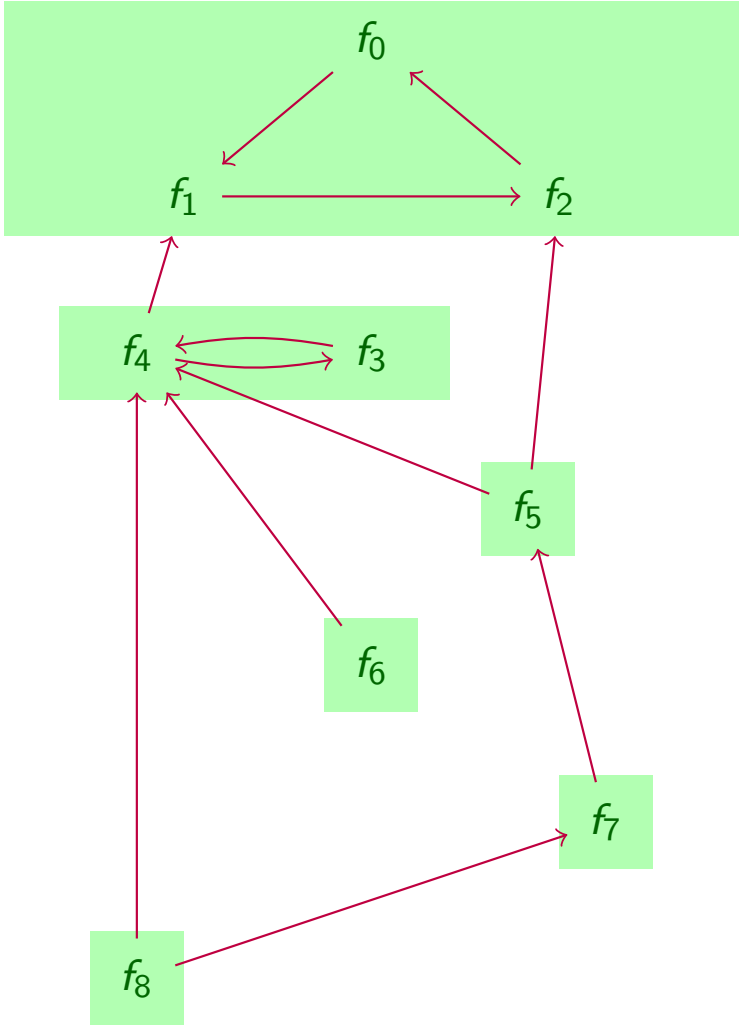
Polymorphic Type Inference Procedure

1. Determine dependencies



Polymorphic Type Inference Procedure

1. Determine dependencies
2. Cluster mutual dependencies



Polymorphic Type Inference Procedure

1. Determine dependencies
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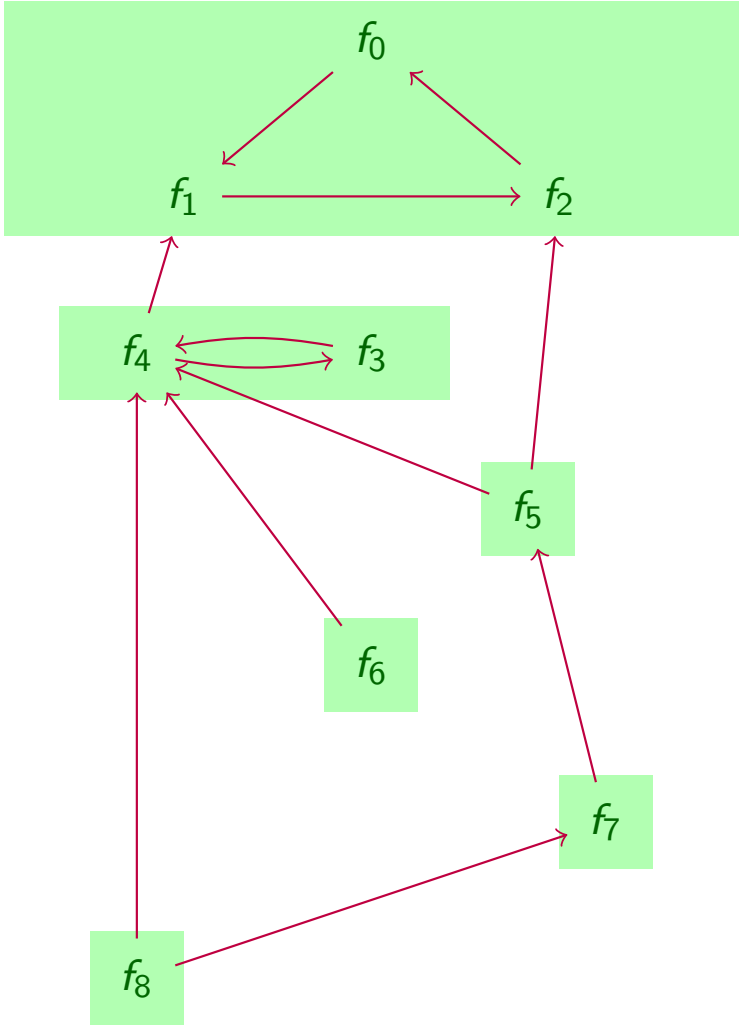
3. Mark clusters: untyped



Polymorphic Type Inference Procedure

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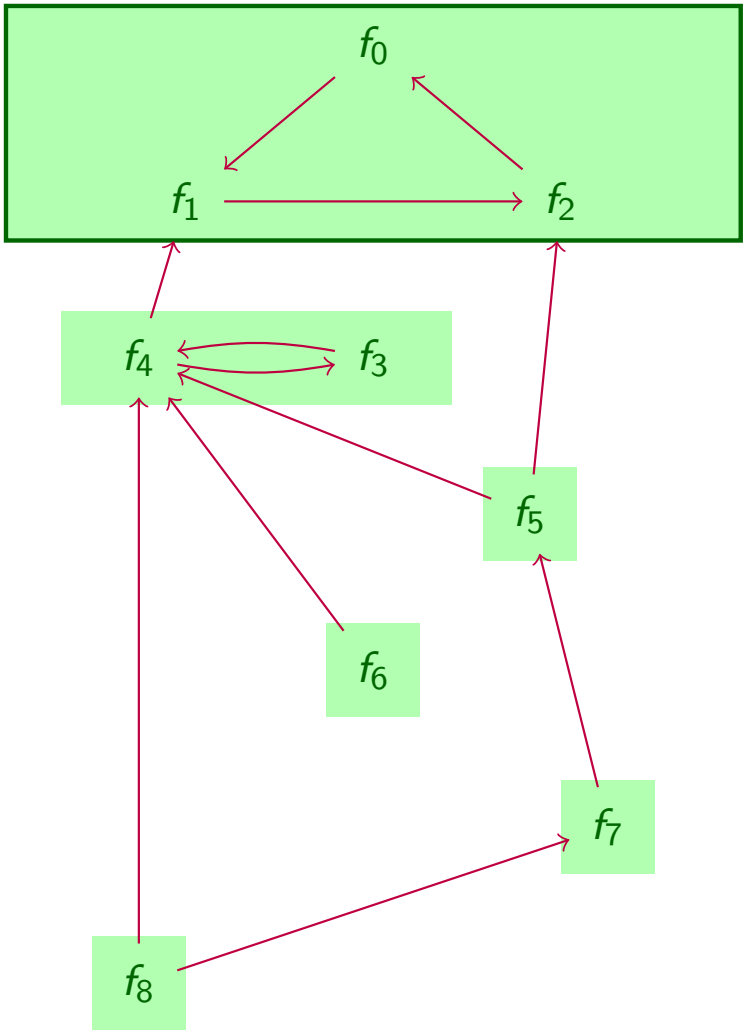
4. While there are untyped clusters:
   4. Pick cluster that has no untyped dependencies



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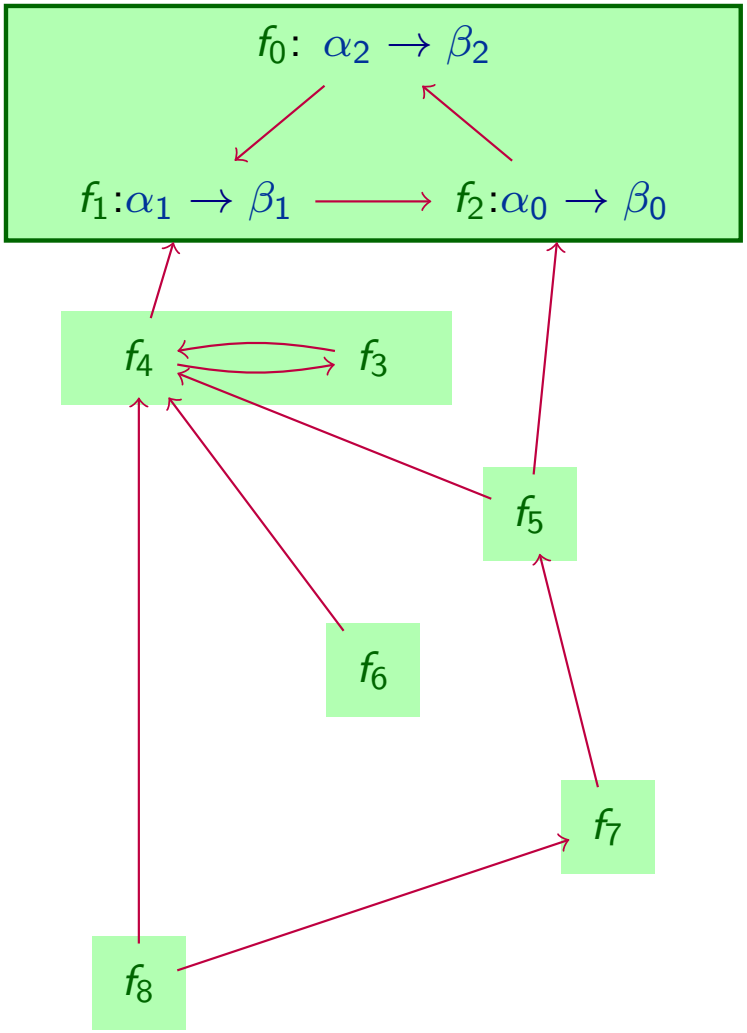

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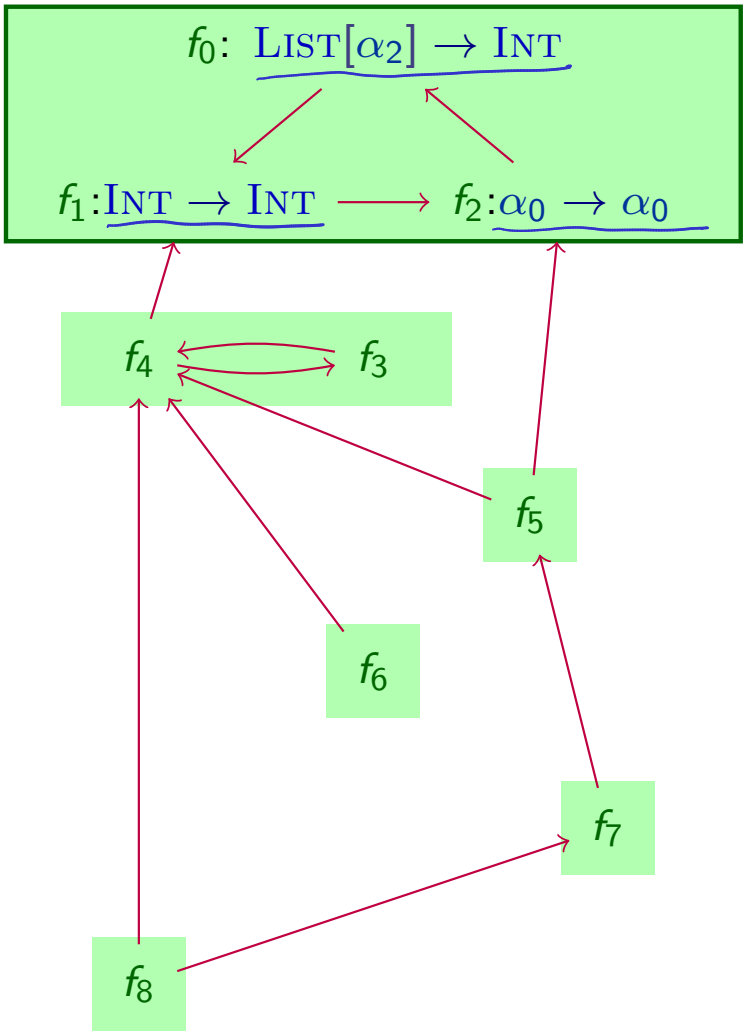

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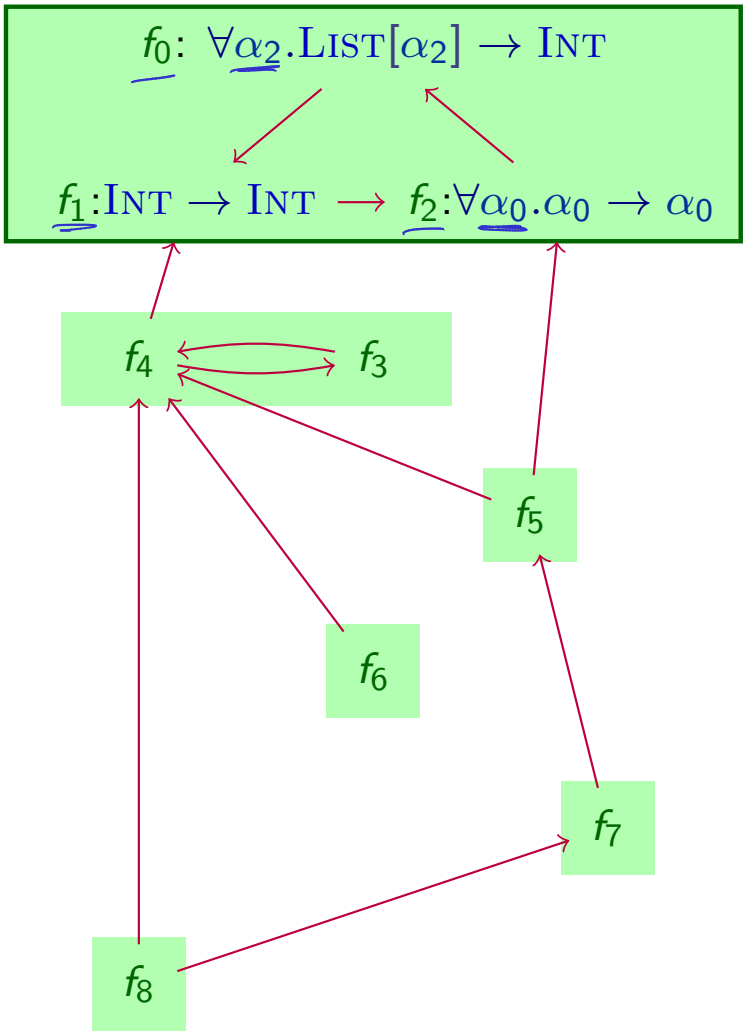

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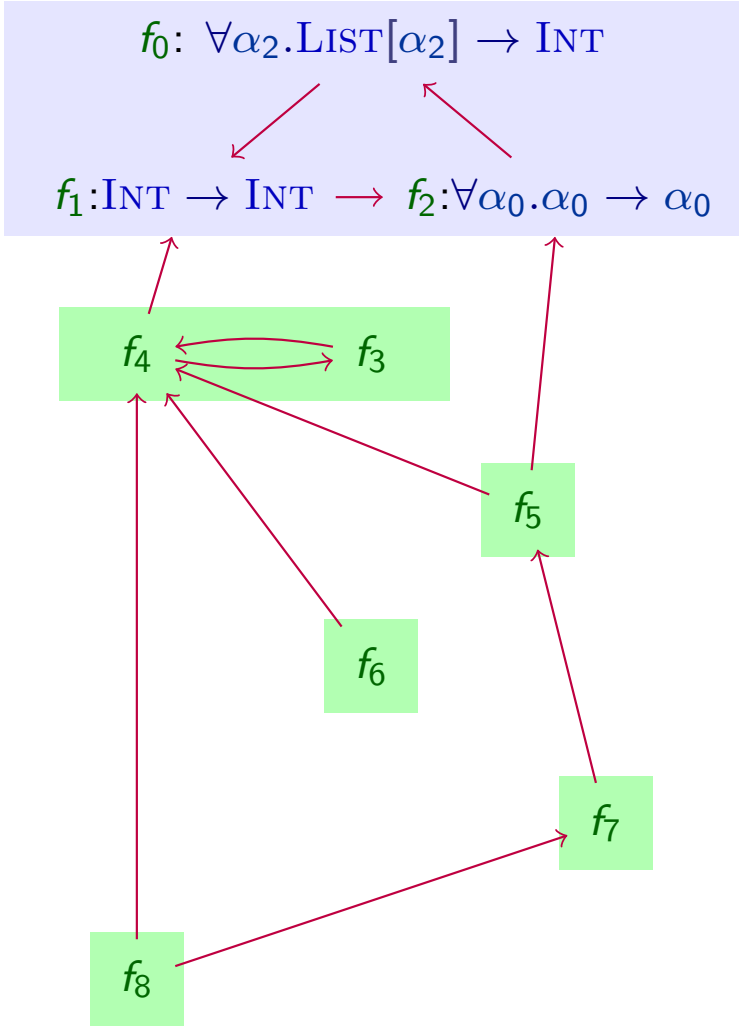
   8. Remove "untyped" mark from cluster



Summary

- **Polymorphic Type Inference** allows generalising types with **Schemes**
- Algorithm:
  - Introduce type variables
  - Systematically apply typing rules to:
    - Generate typings
    - Generate type equality constraints
  - Unify equality constraints 'at the right time'
  - Abstract over free type variables ($\forall$) to introduce schemes
- Must analyse **Dependencies** between definitions
- The 'right time' to unify / abstract:
  - Have finished all dependencies?
  - Can unify all constraints within mutual dependency cluster
- Limitations:
  - Does not handle "inner functions"
    (See Damas-Hindley-Milner, Algorithms $\mathcal{W}$ / $\mathcal{J}$ if interested)
  - Does not handle subtypes

fun f(..) ..
fun g(..) ..

Review: Types for Program Analysis

$$\mathsf{p} : \tau$$

- Types summarise properties of programs
- - Computational results
  - Side effects
  - Dependencies

  ...

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  - *Check* types
  - *Infer* types

Review: Types for Program Analysis

$$\mathsf{p} : \tau$$

- Types summarise properties of programs
- - Computational results
  - Side effects
  - Dependencies

  ...
- *Type Analysis* processes programs recursively to
  - *Check* types
  - *Infer* types
- *Typing Rules*:

$$\frac{e_1 : \textsc{Int} \quad e_2 : \textsc{Int}}{e_1 \ \mathsf{plus} \ e_2 : \textsc{Int}} \ (\textit{t-plus})$$

Parametric and Polymorphic Types

- *Polymorphism*: `p` : τ may have many solutions for τ.
  - A *Principal Type* for `p` summarises all τ.
- *Parametric Types* use *Type Variables* to form principal types
- Generic code creates a challenge:

```
fun f(x) = return x;
```

- *Type Schemes* make the type of `f` reusable:

$$\texttt{f} : \forall \alpha . \alpha \rightarrow \alpha$$

  - Construct type schemes by *Abstraction*
  - *Instantiate* type schemes with *fresh type variables* on demand:

```
f(1)     :  int
f("x")   :  string
```

Example: Dependency Analysis (1/2)

Simplified TEAL-0:

$$
\begin{array}{lcl}
\mathit{module} & ::= & \langle \mathit{decl} \rangle\star \\
\\
\mathit{decl} & ::= & \text{fun } \mathit{id} \text{ (} \mathit{id} \text{) = } \langle \mathit{stmt} \rangle \\
\\
\mathit{expr} & ::= & \mathit{id} \text{ (} \langle \mathit{expr} \rangle \text{)} \\
 & \mid & \mathit{int} \\
 & \mid & \mathit{id} \\
\\
\mathit{block} & ::= & \text{\{ } \langle \mathit{stmt} \rangle\star \text{ \}} \\
\\
\mathit{stmt} & ::= & \text{if } \langle \mathit{expr} \rangle \ \langle \mathit{block} \rangle \text{ else } \langle \mathit{block} \rangle \\
 & \mid & \text{return } \langle \mathit{expr} \rangle \text{ ;} \\
 & \mid & \langle \mathit{block} \rangle
\end{array}
$$

Goal: Let's build a dependency analysis

Example: Dependency Analysis (2/2)

- Type:

$$\frac{}{\underline{f}(e) :} \qquad \frac{v \in int}{v :} \qquad \frac{\underline{x} \in id}{\underline{x} :}$$

Example: Dependency Analysis (2/2)

- Type: *Set of direct dependencies* (invoked functions)

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Example: Dependency Analysis (2/2)

- Type: *Set of direct dependencies* (invoked functions)

$$\frac{}{\underline{f}(e) :} \qquad \frac{v \in int}{v : \emptyset} \qquad \frac{\underline{x} \in id}{\underline{x} : \emptyset}$$

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$$\frac{e : D}{\underline{f}(e) : D \cup \{\underline{f}\}} \qquad \frac{v \in int}{v : \emptyset} \qquad \frac{\underline{x} \in id}{\underline{x} : \emptyset}$$

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$$\frac{}{\textbf{if } e \textbf{ then } b_2 \textbf{ else } b_3 :}$$

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$$\frac{s_i : D_i \text{ for all } i \in \{1, \ldots, n\}}{\{\ s_1 \ \ldots \ s_n\ \} : \bigcup_{i \in \{1,\ldots,n\}} D_i}$$

Building a Program Analysis