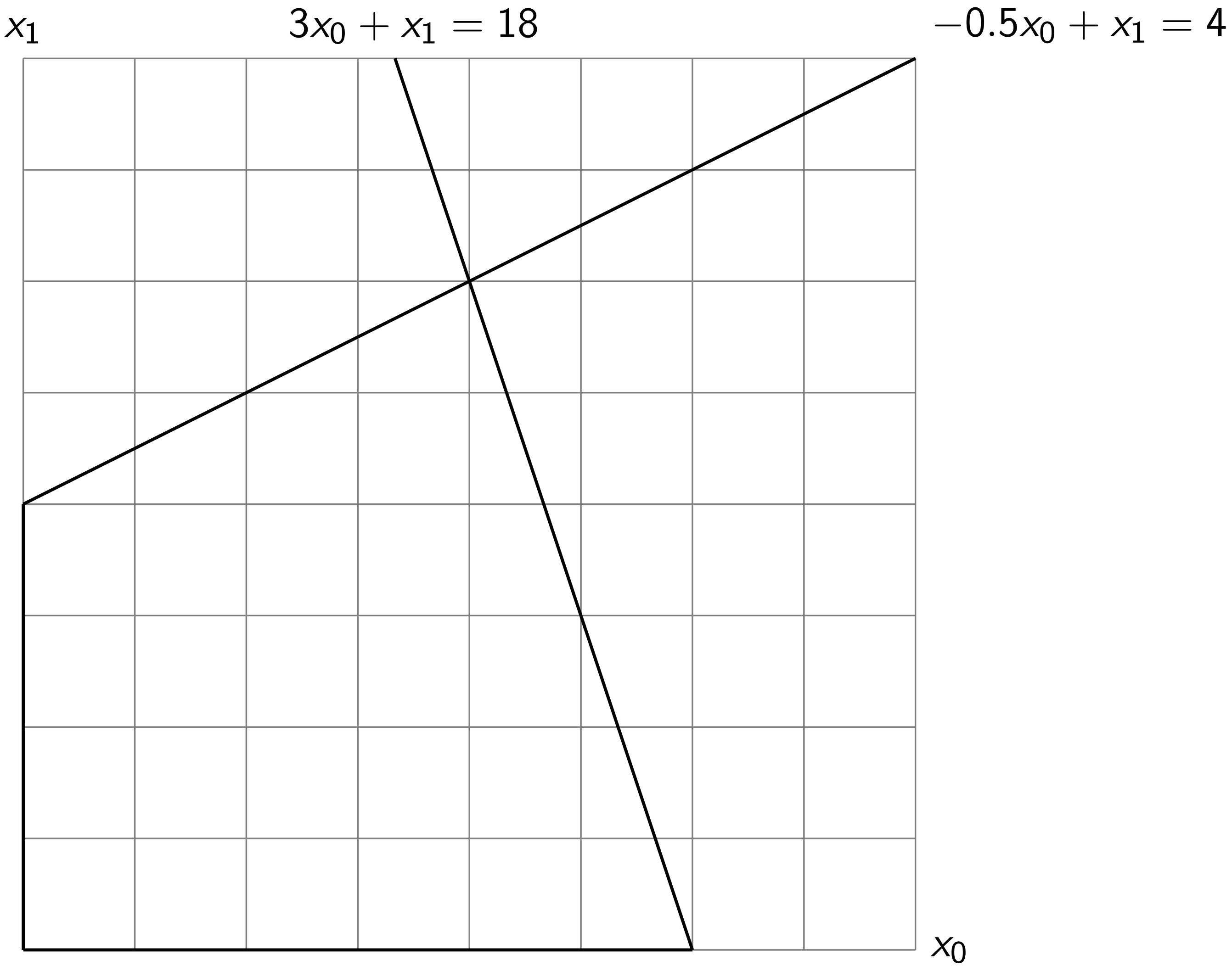
Contents Lecture 2

- Course lab theme and project: integer program solver
- You will implement this during labs 1 — 4
- The labs have other contents as well, such as
    - the gdb debugger
    - Google Sanitizer
    - Valgrind
    - operf, gprof, gcov
- Lab 5: POWER8 pipeline simulator
- Lab 6: optimizing compilers (gcc, clang, and ibm's and nvidia's)

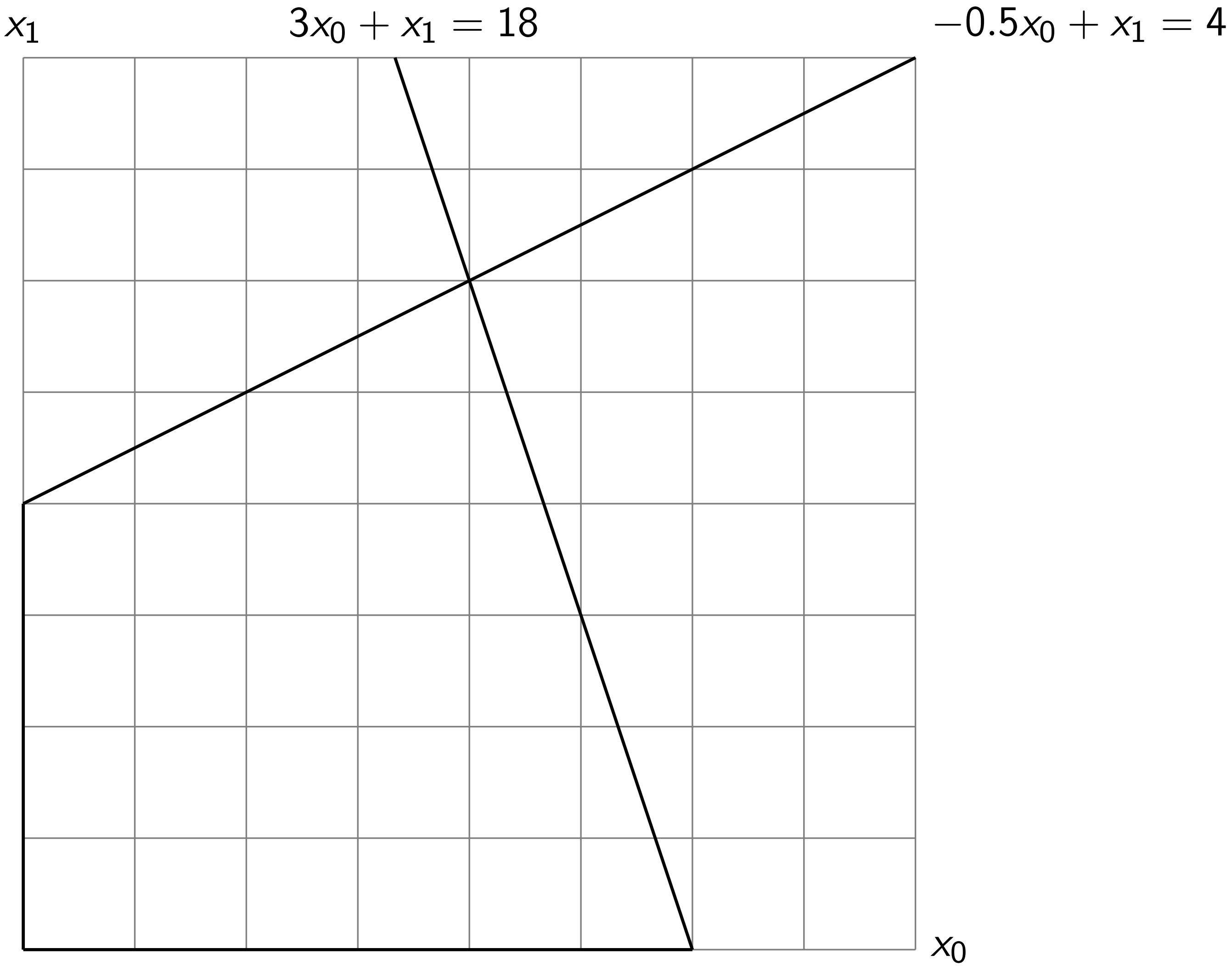
Linear program: maximize a linear function in a region

- a region defined by lines including $x_i \geq 0$, i.e. linear constraints, and
- an objective function such as $\max z = x_0 + 2x_1$



Solving a linear program

- Find $x_i \in \mathbb{R}$ which maximizes z
- Here x_i are called decision variables



Our example

- Objective function and linear constraints using $\leq$

$$\max \quad z \quad = x_0 + 2x_1$$

$$\begin{array}{rcccc} -0.5x_0 & + & x_1 & \leq & 4 \\ 3x_0 & + & x_1 & \leq & 18. \end{array}$$

- Also: implicitly $x_i \geq 0$
- We can use e.g. $x_i \geq 4$ or $x_i = 5$ but they can be rewritten to use $\leq$

Linear programs

- $$\max \quad z = c_0x_0 + c_1x_1 + \ldots + c_{n-1}x_{n-1}$$

$$\begin{array}{rcc} a_{0,0}x_0 + a_{0,1}x_1 + \ldots + a_{0,n-1} & \leq & b_0 \\ a_{1,0}x_0 + a_{1,1}x_1 + \ldots + a_{1,n-1} & \leq & b_1 \\ \ldots & & \\ a_{m-1,0}x_0 + a_{m-1,1}x_1 + \ldots + a_{m-1,n-1} & \leq & b_{m-1} \\ & & \\ x_0, x_1, \ldots, x_{n-1} & \geq & 0 \end{array}$$

- or simpler as

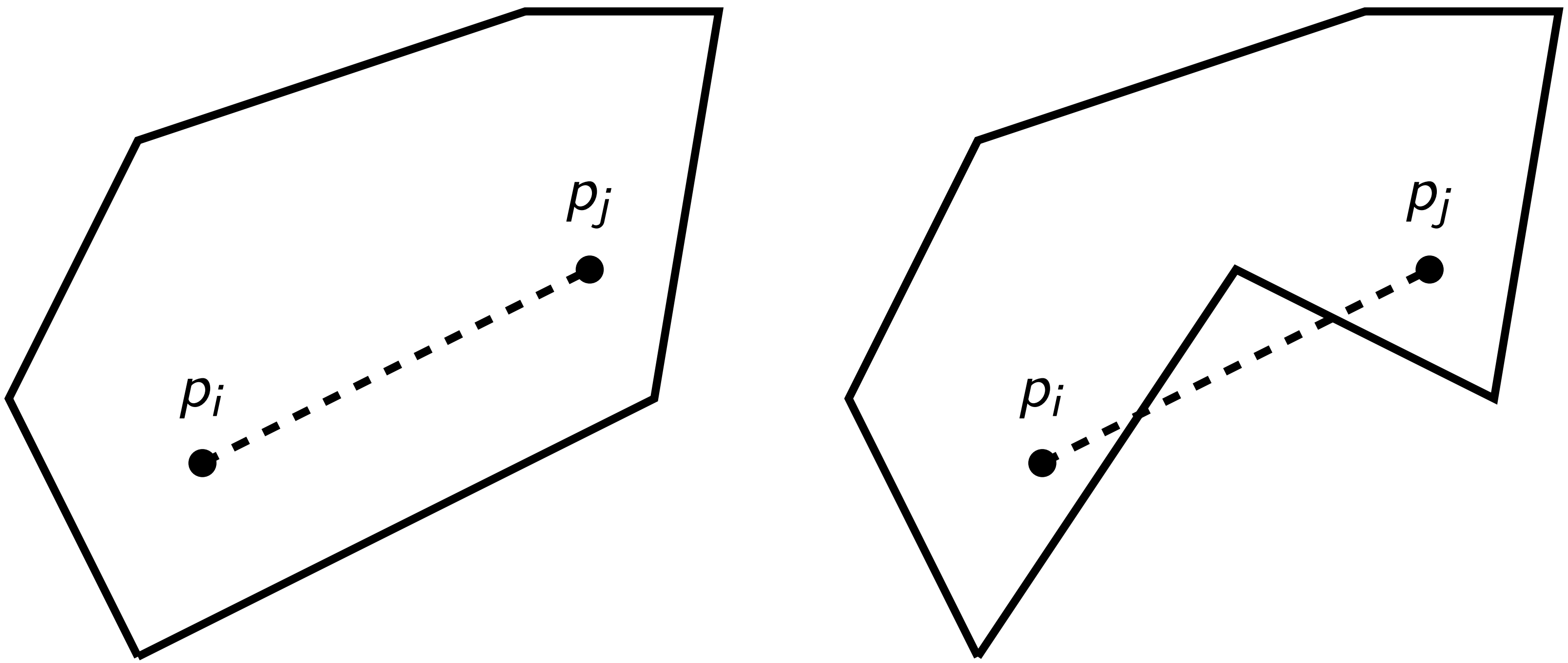
$$\begin{array}{lccc} \max & z = \mathsf{cx} & & \\ & \mathsf{Ax} & \leq & \mathsf{b} \\ & \mathsf{x} & \geq & 0. \end{array}$$

Solutions

- Each constraint defines a halfplane in n dimensions.
- The intersection of these halfplanes defines the **feasible region**, P, with **feasible solutions** $\mathrm{x} \in \mathrm{P}$.
- The feasible region is convex, and a point where halfplanes intersect is called a **vertex**.
- A linear program is either:
  - **infeasible** when P is empty,
  - **unbounded** when no finite solution exists, or
  - **feasible**, in which case we search for an optimal solution $\mathrm{x}^* \in \mathrm{P}$ which maximizes z.
- There may exist more than one optimal solution.

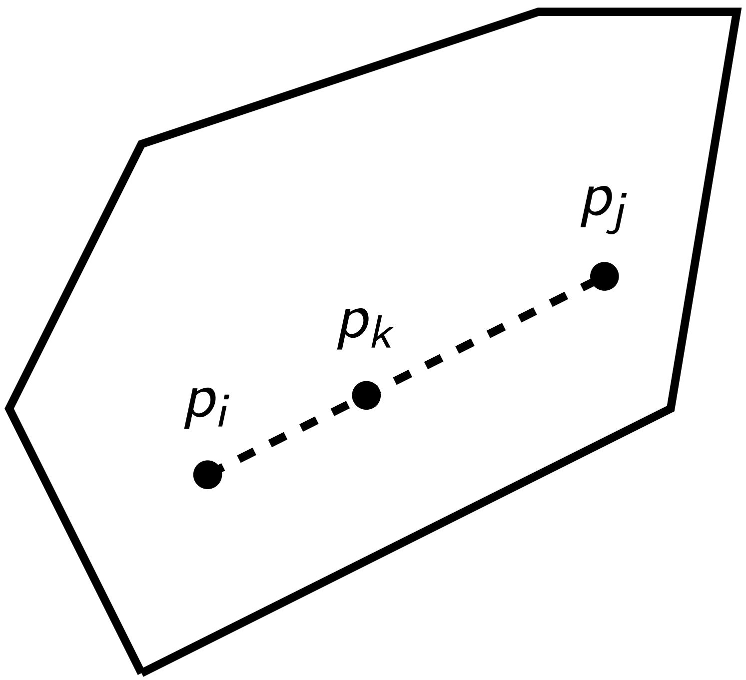
Solutions

- Each constraint defines a halfplane in n dimensions.
- The intersection of these halfplanes defines the **feasible region**, P, with **feasible solutions** $x \in P$.
- The feasible region is convex, and a point where halfplanes intersect is called a **vertex**.
- A non-convex region cannot be an intersection of halfplanes.



Consequence of P being a convex region

- Assume p_k is on a line segment through p_i and p_j
- We can write a point as $p_k = \lambda \cdot p_i + (1 - \lambda) \cdot p_j$, with $0 \leq \lambda \leq 1$
- So that p_k is in the region
- Here $\lambda = 0.4$



Local and global optimal solutions

- We denote by $z(\mathrm{x})$ the value of the objective function z at point x.
- A solution x is a **local optimum** for $z(\mathrm{x})$ if there exists an $\epsilon > 0$ such that $z(\mathrm{x}) \geq z(\mathrm{y})$ for all $\mathrm{y} \in \mathrm{P}$ with $||x - y|| \leq \epsilon$.

Theorem

A local optimum of a linear program is also a global optimum.

Theorem

For a bounded feasible linear program with feasible region P, at least one vertex is an optimal solution.

- So we only need to check z in the vertices and not the inner part of the region.

A local optimum of a linear program is also a global optimum

Proof.

- $z(u)$ is a linear objective function
- Assume $z(u) > z(v)$ for all v at most ϵ from u
- Let w be any point in P, possibly far away from u and v
- $v = \lambda u + (1 - \lambda)w$ with $0 \leq \lambda \leq 1$
- $z(u) \geq z(v) = z(\lambda u + (1 - \lambda)w) = \lambda z(u) + (1 - \lambda)z(w)$
- So $z(u) - \lambda z(u) \geq (1 - \lambda)z(w)$
- And $z(u)(1 - \lambda) \geq (1 - \lambda)z(w)$
- $0 \leq \lambda \leq 1$, so $z(u) \geq z(w)$

$\square$

At least one vertex is an optimal solution

Proof.

- Let the feasible region P have k vertices: $\mathrm{x}^0, \mathrm{x}^1, ..., \mathrm{x}^{\mathrm{k}-1}$
- Let z^* be the maximum value in any vertex: $z^* = \max\{\mathrm{cx}^{\mathrm{i}}, 0 \leq i < k\}$
- Every point w in P can be written as a linear combination of the vertices: $w = \sum_{i=0}^{k-1} \lambda_i \mathrm{x}^{\mathrm{j}}$ with $\sum_{i=0}^{k-1} \lambda_i = 1$
- Let w be any point in P
- $z(\mathrm{w}) = \mathrm{cw} = \mathrm{c} \sum \lambda_i \mathrm{x}^{\mathrm{i}} = \sum \lambda_i (\mathrm{cx}^{\mathrm{i}}) \leq \sum \lambda_i z^* = z^* \sum \lambda_i = z^*$.

$\square$

Slack form

$$
\begin{aligned}
max \quad & cx \\
x_{n+0} \quad &= \quad b_0 - \textstyle\sum_{j=0}^{n-1} a_{0,j}x_j \\
x_{n+1} \quad &= \quad b_1 - \textstyle\sum_{j=0}^{n-1} a_{1,j}x_j \\
&\dots \\
x_{n+m-1} \quad &= \quad b_{m-1} - \textstyle\sum_{j=0}^{n-1} a_{m-1,j}x_j \\
x_i \quad &\geq \quad 0 \qquad\qquad 0 \leq i \leq n+m-1
\end{aligned} \tag{1}
$$

- The variables on the left hand side are called **basic variable** and occur only once, i.e. neither in any sum on the right hand side, nor in the objective function.
- The other variables are called **nonbasic variables**.

Slack form of our example

- We start with

$$\max \quad z \quad = x_0 + 2x_1$$

$$\begin{array}{cccccc} -0.5x_0 & + & x_1 & \leq & 4 \\ 3x_0 & + & x_1 & \leq & 18 \end{array}$$

- and then introduce two new variables, one for each constraint, and write it on slack form:

$$\max \quad z \quad = x_0 + 2x_1 + y$$

$$\begin{array}{cccccccccc} x_2 & = & 4 & - & (& -0.5x_0 & + & x_1 &) \\ x_3 & = & 18 & - & (& 3x_0 & + & x_1 &). \end{array}$$

- All $x_i \geq 0$ and y is initially zero
- We rewrite the problem until all coefficients in the objective function become negative, and set all nonbasic variables to zero

Entering and leaving basic variables

- Select a nonbasic variable with positive c_i coefficient
- We take nonbasic variable x_0 as the so called **entering basic variable**

$$\max \quad z \quad = x_0 + 2x_1 + y$$

$$\begin{array}{ccccccccc} x_2 & = & 4 & - & (& -0.5x_0 & + & x_1 &) \\ x_3 & = & 18 & - & (& 3x_0 & + & x_1 &). \end{array}$$

- Since c_0 is positive, we want to increase x_0 as much as possible
- The basic variables can limit how much x_0 may be increased (if there is no restriction, then the linear program is unbounded)
- x_3 restricts increasing x_0 to at most 6.
- Therefore we select x_3 as the so called **leaving basic variable**.

Rewritten linear program

- We rewrite the linear program by letting the entering and leaving basic variables switch roles.
- This is a tedious but simple algebraic manipulation
- Do this by hand at least once

$$\max \quad z \quad = -0.333x_3 + 1.667x_1 + 6$$

$$\begin{array}{ccccccccc} x_2 & = & 7 & - & (& 0.167x_3 & + & 1.167x_1 &) \\ x_0 & = & 6 & - & (& 0.333x_3 & + & 0.333x_1 &) \end{array}$$

- Next we must select x_1 as entering basic variable
- x_2 is restricted by $7 - 1.167x_1 \geq 0$
- x_0 is restricted by $6 - 0.333x_1 \geq 0$
- x_2 is most restricted and becomes the leaving basic variable

Solution

- All c_i are negative so z cannot be increased with positive values of the nonbasic variables.
- By setting the nonbasic variables to zero, the maximum becomes 16 in $x = (4, 6)$ which indeed is a vertex.

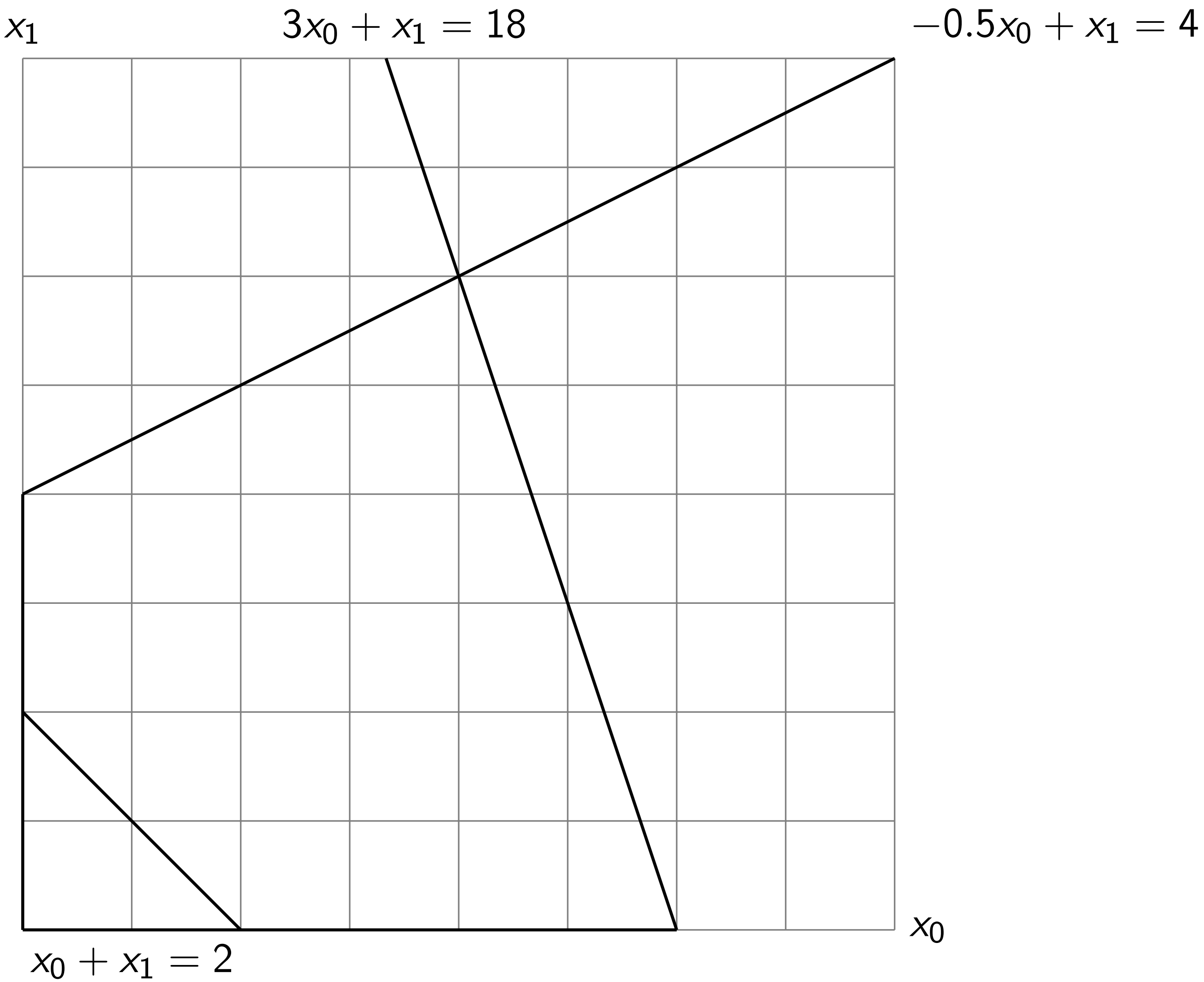
$$\max \quad z \quad = -0.6x_3 - 1.4x_2 + 16$$

$$\begin{array}{ccccccccc} x_1 & = & 6 & - & (& 0.1x_3 & + & 0.9x_2 &) \\ x_0 & = & 4 & - & (& 0.3x_3 & - & 0.3x_2 &) \end{array}$$

- Summary: we start in a vertex and then go to a neighboring vertex until all coefficients are negative, which gives the optimal solution.
- It was an open problem but George Dantzig was late for a lecture at Berkeley and mistook it for a home assignment (he got a PhD for it).

A problem: $(0, 0)$ not in P

- $x_0 + x_1 = 2$ here means $x_0 + x_1 \geq 2$, i.e. $-x_0 - x_1 \leq -2$, i.e. $b_2 = -2$



Finding a start vertex when there is a negative b_i

- Let our original linear program be P_0
- If some b_i is negative the point 0 is not in the feasible region
- We create a new problem P_1 to find a start vertex for P_0
- Add a new nonbasic variable x_{n+m}
- Start with P_0 and subtract x_{n+m} from each constraint:

$$a_{i,0}x_0 + a_{i,1}x_1 + \ldots + a_{i,n-1} - x_{n+m} \quad \leq \quad b_i$$

- Use the objective function $z_1 = -x_{n+m}$
- We do a pivot on P_1 with x_{m+n} as entering basic variable and x_k with most negative b_k as leaving basic variable
- This gives us b $\geq$ 0, so P_1 can be solved using the simplex algorithm,
- If P_1 has optimal value 0 then $x_{n+m} = 0$ and by removing x_{m+n} from this solution, we have a start vertex for P_0

Integer programming

- Integer programming is similar to linear programming with the extra condition that $x_i \in \mathbb{N}$.
- Some problems including this have no efficient algorithms
- One bad "method" to solve problems is to enumerate all solutions
- This does not sound good though
- We will use the algorithm design paradigm branch-and-bound to solve integer programs (not all since integer programming is NP-complete)

Branch

- A **relaxation** makes a problem simpler (by solving another problem)
- For integer programming we solve the corresponding linear program, i.e. relaxing the integer requirement on the solution.
- Suppose we have an integer program and give it to the Simplex algorithm and $x_k \notin \mathbb{N}$
- Assume the Simplex algorithm assigns $x_k = u$
- We can then branch by creating two new linear programs:
  - one with the additional constraint constraint $x_k \leq \lfloor u \rfloor$, and
  - another with the additional constraint $x_k \geq \lceil u \rceil$.
- Each new problem is solved directly with the Simplex algorithm
- If it has an integer solution we can limit the search tree (bound)
- If it has a non-integer solution and it is better than best the integer solution we put it in the queue

Bound

- If the Simplex algorithm found an integer solution we do the following
- We check if this is the best integer solution found so far, and remember it in that case
- We remove from the queue all unexplored linear programs whose optimal value is less than the value of the integer solution we just found

Pseudo code and project

- The distributed pseudo code is in Appendix B in the book printed 2020 and the course Tresorit directory
- It is as simple possible
- It is your task to translate it to C and then to optimize it.
- The only requirement is that it should work for 20 problems (there are thousands problems).
- Not even commercial programs work perfectly
- At `forsete.cs.lth.se` you will be able to upload your C code
- Forsete is an automatic grader and will give you a score
- The score determines who win a coffee mug and does not affect your grade
- You are **not** allowed to use any code you have not written yourself, except functions from the C Standard library.