# Programming language theory

## Solutions

**1** Let $z_1 = \mathcal{A}[\![a_1]\!]\sigma$ and $z_2 = \mathcal{A}[\![a_2]\!]\sigma$ in

$$\mathcal{A}[\![a_1 + a_2]\!]\sigma \triangleq \begin{cases} z_1 + z_2 & \text{if } z_1 \neq \perp \text{ and } z_2 \neq \perp \\ \perp & \text{otherwise} \end{cases}$$

$$\mathcal{A}[\![a_1 / a_2]\!]\sigma \triangleq \begin{cases} z_1 / z_2 & \text{if } z_1 \neq \perp,\ z_2 \neq \perp \text{ and } z_2 \neq 0 \\ \perp & \text{otherwise} \end{cases}$$

The definitions for $\star$ and $-$ are analogous and the definitions for $n$ and $x$ are unchanged.

**2**

$$\frac{\langle\, S_i, \sigma \,\rangle \rightarrow \sigma'}{\langle\, \texttt{if}\ b_1 \rightarrow S_1 \mid b_2 \rightarrow S_2\ \texttt{fi}, \sigma \,\rangle \rightarrow \sigma'} \qquad \text{if } \mathcal{B}[\![b_i]\!]\sigma, \quad i = 1, 2.$$

**3**

$$\mathcal{S}_{ds} \in \text{Stm} \rightarrow 2^{\text{State}} \rightarrow 2^{\text{State}}$$

$$\text{Let } \Sigma \in 2^{\text{State}}$$

$$\mathcal{S}_{ds}[\![x := a]\!]\Sigma \triangleq \{\sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma] \mid \sigma \in \Sigma\}$$

$$\mathcal{S}_{ds}[\![S_1\ \texttt{or}\ S_2]\!]\Sigma \triangleq \mathcal{S}_{ds}[\![S_1]\!]\Sigma \cup \mathcal{S}_{ds}[\![S_2]\!]\Sigma$$

$$\mathcal{S}_{ds}[\![S_1; S_2]\!]\Sigma \triangleq \mathcal{S}_{ds}[\![S_2]\!](\mathcal{S}_{ds}[\![S_1]\!]\Sigma)$$

**4** $$\frac{p \rightarrow q \qquad q \rightarrow p}{p \leftrightarrow q}[\leftrightarrow_I] \qquad \frac{p \leftrightarrow q}{p \rightarrow q}[\leftrightarrow_E] \qquad \frac{p \leftrightarrow q}{q \rightarrow p}[\leftrightarrow_E]$$

**5** Construct a fix point equation:

$$\text{leq}\ n\ m = \textit{if}\ n == 0 \text{ then } \mathbf{tt} \text{ else if } m == 0 \text{ then } \mathbf{ff} \text{ else } \text{leq}(n-1)\ (m-1)$$

$$\text{leq} = \lambda n.\lambda m.\textit{if}\ n == 0 \text{ then } \mathbf{tt} \text{ else if } m == 0 \text{ then } \mathbf{ff} \text{ else } \text{leq}(n-1)\ (m-1)$$

$$\text{leq} = f\ \text{leq}$$

where

$$f \triangleq (\lambda g.\lambda n.\lambda m.\textit{if}\ n == 0 \text{ then } \mathbf{tt} \text{ else if } m == 0 \text{ then } \mathbf{ff} \text{ else } g\ (n-1)\ (m-1))$$

Translating to lambda notation

$$F \triangleq \lambda g.\lambda n.\lambda m.\text{isZero}\ n\ \mathsf{T}\ (\text{isZero}\ m\ \mathsf{F}\ (g\ (\text{Pred}\ n)(\text{Pred}\ m)))$$

Using the fix point combinator

$$\text{Leq} \triangleq Y\ F$$

**6** **a.** The least element is $\mathbb{N}$ since $S \subseteq \mathbb{N}$ for all $S$.

**b.** $\bigsqcup\{\{0,1\},\{1,2\}\} = \{1\}$

**c.** $\{S_i \mid i \in \mathbb{N}\}$ where $S_i \triangleq \{n \mid n \leq i \vee i \text{ is even}\}$.

**d.** Assume that $A \sqsubseteq_D B$. This means that $B \subseteq A$. By the definition of compl it follows that $\text{compl}(A) \subseteq \text{compl}(B)$, i.e. $\text{compl}(A) \sqsubseteq_{D'} \text{compl}(B)$.

**e.** The least element, $\mathbb{N}$ cannot be removed. Any set with one element, e.g. $\{0\}$, can be removed. I am thankful to K-M Perfekt for this observation. Remark: Then the least upper bound will not always be the intersection, $\bigsqcup\{\{0,1\},\{0,2\}\} = \emptyset$.