Solutions

1 Below, a stands for append, c for cons and e for empty.

$$\texttt{unify}(a(c(1,e),c(2,(c(3,e))),L),a(c(X,Xs),Ys,c(X,Zs)))$$
$$\quad \texttt{unify}(c(1,e),c(X,Xs))$$
$$\quad\quad \texttt{unify}(1,X)$$
$$\quad\quad\quad \texttt{unify}(X,1) \qquad X \mapsto 1$$
$$\quad\quad \texttt{unify}(e,Xs)$$
$$\quad\quad\quad \texttt{unify}(Xs,e) \qquad Xs \mapsto e$$
$$\quad \texttt{unify}(c(2,(c(3,e))),Ys)$$
$$\quad\quad \texttt{unify}(Ys,c(2,(c(3,e)))) \qquad Ys \mapsto c(2,(c(3,e)))$$
$$\quad \texttt{unify}(L,c(X,Zs)) \qquad L \mapsto c(1,Zs)$$

2

$$male(oskar)$$
$$\quad \texttt{unify}(male(oskar),male(gustav)) \qquad fails$$
$$\quad father(oskar,Y)$$
$$\quad\quad \texttt{unify}(father(oskar,Y),father(gustav,eva)) \qquad fails$$
$$\quad\quad \texttt{unify}(father(oskar,Y),father(gustav,lena)) \qquad fails$$
$$\quad\quad \texttt{unify}(father(oskar,Y),father(oskar,gustav)) \qquad success$$

3 $\mathcal{S}_{cs}[\![\texttt{halt}]\!]\,c\,\sigma = \sigma$.

4 Assume that $\mathcal{S}'_{cs}[\![S_i]\!]c\,\sigma = c(\mathcal{S}_{ds}[\![S_i]\!]\sigma)$ for $i = 1, 2$. Then

$$\mathcal{S}'_{cs}[\![\texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2]\!]c\,\sigma =$$
$$(\mathcal{B}[\![b]\!]\sigma\ ?\ \mathcal{S}'_{cs}[\![S_1]\!]c\,\sigma\ :\ \mathcal{S}'_{cs}[\![S_2]\!]c\,\sigma) =$$
$$(\mathcal{B}[\![b]\!]\sigma\ ?\ c(\mathcal{S}_{ds}[\![S_1]\!]\sigma)\ :\ c(\mathcal{S}_{ds}[\![S_2]\!]\sigma)) =$$
$$c(\mathcal{B}[\![b]\!]\sigma\ ?\ \mathcal{S}_{ds}[\![S_1]\!]\sigma\ :\ \mathcal{S}_{ds}[\![S_2]\!]\sigma) =$$
$$c(\mathcal{S}_{ds}[\![\texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2]\!]\sigma)$$

5 By the assignment axiom

$$\{s + (x-1)*y = p\}\texttt{x} := \texttt{x} - \texttt{1}\{s + x*y = p\} \text{ and}$$
$$\{s + y + (x-1)*y = p\}\texttt{s} := \texttt{s} + \texttt{y}\{s + (x-1)*y = p\}$$

By the rule for composition and the consequence rule it follows that

$$\{\neg(x = 0) \land s + x*y = p\}\texttt{s} := \texttt{s} + \texttt{y};\texttt{x} := \texttt{x} - \texttt{1}\{s + x*y = p\}$$

6 The statement should be equivalent to

$$S;\ \ \texttt{while}\ \neg b\ \texttt{do}\ S$$

Consider the following proof tree

$$\cfrac{\{P\}S;\{Q\} \qquad \cfrac{\{Q \wedge \neg b\}S\{Q\}}{\{Q\}\ \texttt{while}\ \neg b\ \texttt{do}\ S\ \{Q \wedge b\}}}{\{P\}S;\ \ \texttt{while}\ \neg b\ \texttt{do}\ S\ \{Q \wedge b\}}$$

to justify the following proof rule

$$\frac{\{P\}S\{Q\} \qquad \{Q \wedge \neg b\}S\{Q\}}{\{P\}\ \texttt{repeat}\ S\ \texttt{until}\ b\ \{Q \wedge b\}}$$

7 $\mathtt{x} * 2^{\mathtt{y}} = 2^{\mathtt{n}}$.

8 Assume that $P \Rightarrow Q$. Base cases:

$\mathcal{S}_{\text{wp}}[\![\texttt{skip}]\!]\text{P} = \text{P}$ and
$\mathcal{S}_{\text{wp}}[\![\texttt{skip}]\!]\text{Q} = \text{Q}$ implies that
$\mathcal{S}_{\text{wp}}[\![\texttt{skip}]\!]\text{P} \Rightarrow \mathcal{S}_{\text{wp}}[\![\texttt{skip}]\!]\text{Q}$

The definition of substitution implies that $(P[x \mapsto a]) \Rightarrow (Q[x \mapsto a])$.

$\mathcal{S}_{\text{wp}}[\![\texttt{x:=a}]\!]\text{P} = \text{P}[x \mapsto a]$ and
$\mathcal{S}_{\text{wp}}[\![\texttt{x:=a}]\!]\text{Q} = \text{Q}[x \mapsto a]$ implies that
$\mathcal{S}_{\text{wp}}[\![\texttt{x:=a}]\!]\text{P} \Rightarrow \mathcal{S}_{\text{wp}}[\![\texttt{x:=a}]\!]\text{Q}$

Inductive cases: Assume that the statement in the problem is true for S_1 and S_2 and all P and Q. Thus

$\mathcal{S}_{\text{wp}}[\![S_2]\!]\text{P} \Rightarrow \mathcal{S}_{\text{wp}}[\![S_2]\!]\text{Q}$ and
$\mathcal{S}_{\text{wp}}[\![S_1]\!](\mathcal{S}_{\text{wp}}[\![S_2]\!]\text{P}) \Rightarrow \mathcal{S}_{\text{wp}}[\![S_1]\!](\mathcal{S}_{\text{wp}}[\![S_2]\!]\text{Q})$

We omit the case for the if statement.

For the `while` statement, assume that $\text{P} \Rightarrow \text{Q}$ implies that $\mathcal{S}_{\text{wp}}[\![S]\!]\text{P} \Rightarrow \mathcal{S}_{\text{wp}}[\![S]\!]\text{Q}$ for all P and Q. It follows that $W_0(\text{P}) \Rightarrow W_0(\text{Q})$ and that $W_i(\text{P}) \Rightarrow W_i(\text{Q})$ implies $W_{i+1}(\text{P}) \Rightarrow W_{i+1}(\text{Q})$ for all i and finally that $\exists n \in \text{N} \,.\, W_n(\text{P}) \Rightarrow \exists n \in \text{N} \,.\, W_n(\text{Q})$.