Solutions

1. $< \texttt{GET-}n : c, e, m > \triangleright < c, m[n] : e, m >$ if $0 < n \leq \text{length } m$

   $< \texttt{PUT-}n : c, z : e, m > \triangleright < c, e, m[n \mapsto z] >$ if $0 < n \leq \text{length } m$

   If we reference an address outside the memory the execution terminates in a nonterminal configuration.

2.

chain	least upper bound	chain	least upper bound
$\emptyset$	$\emptyset$	$\{\{1\}, \{0,1\}\}$	$\{0,1\}$
$\{\emptyset\}$	$\emptyset$	$\{\emptyset, \{0\}\}$	$\{0\}$
$\{\{0\}\}$	$\{0\}$	$\{\emptyset, \{1\}\}$	$\{1\}$
$\{\{1\}\}$	$\{1\}$	$\{\emptyset, \{0,1\}\}$	$\{0,1\}$
$\{\{0,1\}\}$	$\{0,1\}$	$\{\emptyset, \{0\}, \{0,1\}\}$	$\{0,1\}$
$\{\{0\}, \{0,1\}\}$	$\{0,1\}$	$\{\emptyset, \{1\}, \{0,1\}\}$	$\{0,1\}$

3. Monotonicity. Assume that $S_0 \subseteq S_1$. Prove that $F(S_0) \subseteq F(S_1)$. Assume that $x \in F(S_0) = S_0 \cup A_0$ then $x \in S_0$ or $x \in A_0$. It follows that $x \in S_1$ or $x \in A_0$. This means that $x \in S_1 \cup A_0$, i.e. $x \in F(S_1)$.

   Continuity. Let C be a chain in 2^A. In $(2^A, \subseteq)$ the least upper bound $\bigsqcup$ is the same thing as $\bigcup$.

   We have to show that $\bigcup F(C) = F(\bigcup C)$. Remember that $F(C) = \{F(c) \mid c \in C\}$. We divide this proof in to parts, $\bigcup F(C) \subseteq F(\bigcup C)$ and $F(\bigcup C) \subseteq \bigcup F(C)$.

   First assume $x \in \bigcup F(C) = \bigcup\{F(c) \mid c \in C\}$. Then there is a $c \in C$ such that $x \in F(c)$. Since $c \subseteq \bigcup C$ and F is monotone it follows that $F(c) \subseteq F(\bigcup C)$. Thus $x \in F(\bigcup C)$. (This is just the proof of Lemma 4.30. We did not use the definition of F!)

   Next assume that $x \in F(\bigcup C)$. Then $x \in (\bigcup C) \cup A_0$, i.e. $x \in (\bigcup C)$ or $x \in A_0$. It follows that $x \in c$ for some $c \in C$ or $x \in A_0$. Thus $x \in c \cup A_0 = F(c)$ for some $c \in C$. We conclude that $x \in \bigcup F(C)$.

4. Assume that $A_1 \subseteq A_2$. We have to show that $\mathcal{P}(A_1) \subseteq \mathcal{P}(A_2)$. Let $x \in \mathcal{P}(A_1)$. Then $x \subseteq A_1$. It follows that $x \subseteq A_2$ and $x \in \mathcal{P}(A_2)$ and $\mathcal{P}$ is monotone.

   $\mathcal{P}$ is not continuous. To see this consider the chain $C \triangleq \{N_n \mid n \in \mathrm{N}\}$, where $N_n \triangleq \{0, \ldots, n-1\}$. Then $\bigcup C = \mathrm{N}$ (which does not belong to C). The set $\mathcal{P}(\bigcup C) = 2^{\mathrm{N}}$ contains all finite and infinite sets of natural numbers. On the other hand all elements of $\bigcup \mathcal{P}(C) = \bigcup\{\mathcal{P}(N_n) \mid n \in \mathrm{N}\}$ are finite sets. So e.g. $\mathrm{N} \in \mathcal{P}(\bigcup C)$ while $\mathrm{N} \notin \bigcup \mathcal{P}(C)$.

5. S will be executed in the state σ giving a new state σ'. If b is true in this new state this will be the effect of the execution. If b is false in the new state the statement has no effect.

6. $\mathcal{S}_{ds}[\![x_0 :=: x_1]\!]\sigma = \sigma[x_0 \mapsto (\sigma\, x_1)][x_1 \mapsto (\sigma\, x_0)]$

7.

$$\mathcal{A} \in \mathbf{Aexp} \to \mathbf{State} \to \mathrm{Z} \times \mathbf{State}$$

$$\mathcal{A}[\![x]\!]\,\sigma \triangleq (\sigma x, \sigma)$$

$$\mathcal{A}[\![x{+}{+}]\!]\,\sigma \triangleq (\sigma x, \sigma[x \mapsto 1 + \sigma\, x])$$

$$\mathcal{A}[\![a_1 + a_2]\!]\,\sigma \triangleq (z_1 + z_2, \sigma_2)$$

where

$$(z_1, \sigma_1) = \mathcal{A}[\![a_1]\!]\sigma$$

$$(z_2, \sigma_2) = \mathcal{A}[\![a_2]\!]\sigma_1$$

The remaining expressions are similar.

8.

$$F g = \lambda\sigma.(\mathcal{B}[\![\texttt{true}]\!]\sigma \ ?\ (g \circ (\lambda\sigma.\sigma))\sigma \ :\ \sigma) = \lambda\sigma.g\sigma = g$$
$$F^0\,\emptyset = \emptyset$$
$$F\,\emptyset = \emptyset$$
$$F^2\,\emptyset = \emptyset$$

It follows that $\bigcup\{F^n\emptyset \mid n \in \mathrm{N}\} = \emptyset$.