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Computer Science
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Programming Language Theory
Lecture notes

8 Statement logic

8.1 Introduction

Statement logic is about computing with the logical operators 'not', 'and', and 'or'. We assume that the reader is familiar with such computations both from mathematics and programming.

We will introduce rules that formalize our way to reason. These rules will have the same form as some rules used to define the semantics of programming languages.

Statement logic is the basis for *predicate logic* and *logic programming.*

8.2 Natural deduction

When you prove something you will use *inference* or *deduction rules* to combine statements that already have been proved or are assumed to be true A common inference rule is *modus ponens* (rule of detachment). If it is known that $P \rightarrow Q$ and that P are true one may *deduce* that Q is true.

Inference rules are sometimes described by a formula. Modus ponens is given by

$$\frac{P \qquad P \rightarrow Q}{Q}\,[MP]$$

Above the line are the *assumptions*, the statements that must be true in order to conclude that the statement below the line, the *conclusion*, to be true. The name of the rule may be indicated near the line. The order of the assumptions is not significant.

When using the rule P and$Q$ must be replaced by expressions of statement logic. The rule is a *schema* and each use of the rule is called an *instance.* We are using capital letters as place holders or variables to be replace by logic expressions. Below two instances of modus ponens are shown.

$$\frac{p \qquad p \rightarrow q}{q}\,[MP] \qquad \frac{(p \wedge q) \qquad (p \wedge q) \rightarrow \neg p}{\neg p}\,[MP]$$

When replacing the the variables with composite expressions they should be enclosed in parentheses which may be removed if this can be done without changing the meaning.

In a mathematical *theory* one must always specify which inference rules may be used in proofs. Often there is also *axioms* which are statements that assumed to be true without proof. The axioms describes the fundamental properties of the objects of the theory. In most theories logic is an integral part of the theory and standard inference rules formalizes the meaning of the logical operators.

We shall present some of the inference rules used in mathematical proofs. This particular formalism is called *natural deduction.* With these rules it is possible to deduce all tautologies. The purpose of this section is to show how one may make deductions in a way that makes it possible to check them by mechanical means, e.g. by using a computer program. If the purpose is just to prove that an expression is a tautology it is easier to mechanically fill in a truth table. If the number of statement letters is large it is sometimes possible to find shortcuts by using the structure of the expression. However, this is not always the case; the general problem belongs to class called NP-complete problems that may be intractable when the number of variables increases.

to every operator there are rules to *introduce* and to *eliminate* the operator. The names of the rules indicate weather an operator is eliminated (E) or introduced (I).

The rules for $\wedge$.

$$\frac{P \wedge Q}{P}\,[\wedge_{E_1}] \qquad \frac{P \wedge Q}{Q}\,[\wedge_{E_2}] \qquad \frac{P \qquad Q}{P \wedge Q}\,[\wedge_I]$$

The rules are easy to understand and accept. The first one states that if we know that $P \wedge Q$ is true we may conclude that P is true, The second rules says the analogue of Q. The third rule states that if we know that P is true and that Q is true then $P \wedge Q$ is true.

The instances of the rules may be combined so that the conclusions of one or several instances become the premises of another. Such a construction is called a *derivation.* A derivation is thus a tree. A derivation describes a logical reasoning where the statements in the leaves are premisses that are sufficient to prove the conclusion at the root.

With the above rules we can construct a derivation that proves that if $p \wedge q$ is true then $q \wedge p$ is true. Since it is easy to see which rule that gas been used in every step the derivation cab be presented without such information. Since the order of the premises is insignificant the third derivation is also valid.

$$\frac{\dfrac{p \wedge q}{q}\,[\wedge_{E_2}] \qquad \dfrac{p \wedge q}{p}\,[\wedge_{E_1}]}{q \wedge p}\,[\wedge_I] \qquad \frac{\dfrac{p \wedge q}{q} \qquad \dfrac{p \wedge q}{p}}{q \wedge p} \qquad \frac{\dfrac{p \wedge q}{p} \qquad \dfrac{p \wedge q}{q}}{q \wedge p}$$

When there is a derivation which uses a set of premises A that proves the conclusion Q we write

$$A \vdash Q$$

This formula is called a *sequent.* Our example shows that $\{ p \wedge q \} \vdash q \wedge p$. If A is the empty set it may be omitted, $\vdash Q$.

Activity 1

Construct a derivation that proves $\{ (p \wedge q) \wedge r \} \vdash p \wedge (q \wedge r)$.

Some rules allows the use of a *hypothetical assumption* in the derivation. The hypothetical assumption is written inside brackets when introduced and stroked out when the rule is used. The assumption is *discharged.* A hypothetical assumption may be used with the inference rules in the same way as other premises.

We meet a hypothetical assumption in the introduction rule for $\rightarrow$ that states that if we can deduce Q from a hypothetical assumption, P, then we may conclude that $P \rightarrow Q$ and at the same time strike out the hypothesis in the derivation.

$$\dfrac{\begin{matrix}[P]\\ \vdots \\ Q\end{matrix}}{P \to Q}\,[\to_I]$$

In the general case the derivation of Q is a tree and it is allowed to introduce the hypothetical assumption zero or more times. Some simple examples using the rule:

$$\dfrac{[p]}{p \to p}\,[\to_I] \qquad \dfrac{\dfrac{[p \land q]}{p}\,[\land_{E_1}]}{p \land q \to p}\,[\to_I] \qquad \dfrac{\dfrac{\dfrac{[p \land q]}{q}\,[\land_{E_2}] \quad \dfrac{[p \land q]}{p}\,[\land_{E_1}]}{q \land p}\,[\land_I]}{p \land q \to q \land p}\,[\to_I]$$

The elimination rule for $\to$ is the same as modus ponens.

$$\dfrac{P \qquad P \to Q}{Q}\,[\to_E]$$

Activity 2

Let us use the rules to prove that $\{\, p \to q, q \to r \,\} \vdash p \to r$. The derivation has been started with a hypothetical assumption. Complete the derivation and indicate which rule has been used in each node.

$$\dfrac{\dfrac{\dfrac{[p] \qquad p \to q}{q}\,[\to_E] \qquad q \to r}{r}\,[\to_E]}{p \to r}\,[\to_I]$$

The rules for $\lor$:

$$\dfrac{P \lor Q \qquad \begin{matrix}[P]\\ \vdots \\ R\end{matrix} \qquad \begin{matrix}[Q]\\ \vdots \\ R\end{matrix}}{R}\,[\lor_E] \qquad \dfrac{P}{P \lor Q}\,[\lor_{I_1}] \qquad \dfrac{Q}{P \lor Q}\,[\lor_{I_2}]$$

The first rule state that if we know that P or Q is true and if we have a derivation of R provided that Q is true, then we may conclude that R is true. The introduction rules are easy to understand.

Activity 3

Show that $\{\, p \lor (q \land r) \,\} \vdash p \lor q$.

$$\dfrac{p \lor (q \land r) \qquad \dfrac{[p]}{p \lor q} \qquad \dfrac{\dfrac{[q \land r]}{q}}{p \lor q}}{p \lor q}$$

The introduction rule for $\neg$ is about *proof by contradiction*. In a proof by contradiction one assumes the negation of the statement one plans to prove and shows that this assumption leads to a *contradiction*, i.e. two statements where one statement is the negation of the other.

$$\dfrac{\begin{array}{c}[P]\\ \vdots \\ R\end{array} \qquad \begin{array}{c}[P]\\ \vdots \\ \neg R\end{array}}{\neg P}\,[\neg_I]$$

As before it is admissible to introduce the hypothetical assumption any number of times. If both R and $\neg R$ have been derived without any hypothetical assumptions then any statement may be concluded.

A simple example using $\neg$ introduction shows that $\vdash \neg(p \wedge \neg p)$.

$$\dfrac{\dfrac{[p \wedge \neg p]}{p} \qquad \dfrac{[p \wedge \neg p]}{\neg p}}{\neg(p \wedge \neg p)}$$

The elimination rule for $\neg$ differs from the other elimination rules as it eliminates two operators in one step:

$$\dfrac{\neg\neg P}{P}\,[\neg_E]$$

Using the inference rules all tautologies may be derived without any undischarged assumptions and vice versa. If we use $\models P$ to mean that P is true for all all values of the identifiers in P then we have

Theorem. $\models P$ om och endast om $\vdash P$. ■

In mathematical theorems and proofs one often use the symbol $\Leftrightarrow$ in this context:

$\models P \Leftrightarrow \vdash P$

The proof is a usual, by induction. We will not prove this theorem.

Some tautologies requires derivations that are astonishingly long. The following derivation of $\vdash p \vee \neg p$ uses three hypothetical assumptions.

Activity 4

Indicate which rule is used in each node.

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]}{p \vee \neg p}\,[\vee_I] \qquad [\neg(p \vee \neg p)]}{\neg p}\,[\neg_I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg p]}{p \vee \neg p}\,[\vee_I] \qquad [\neg(p \vee \neg p)]}{\neg\neg p}\,[\neg_I]}{p}\,[\neg_E]}{}}{\neg\neg(p \vee \neg p)}\,[\neg_I]}{p \vee \neg p}\,[\neg_E]$$

When the derivation is complete all hypothetical assumptions have been discharged. We have proved $\vdash p \vee \neg p$.

It is easy to understand that we could replace p by any statement logic expression. We could add a new inference rule

$$\frac{}{P \vee \neg P}$$

without changing the set of statements that could be deerived, but some derivations would be shorter. Such rules are named *derived*. In mathematical proofs many sch derived rules are used.

This derived rule differs from the other rules in the respect that it has no premises. An axiom may be seen as an inference rule without premises.

8.3 Theories

A mathematical *theory* is defined by a number of axioms and inference rules. Using these *theorems* may be derived. An axiom is a statement that is assumed to be true without proof. The axiom describes the fundamental properties of the elements of the theory. In the geometrical theory of Euclides *points*, *lines*, and *planes* are some of the fundamental elements.

9 Referenser

1. D. Prawitz: *Natural Deduction. A Proof-Theoretical Study*, Almqvist & Wiksell, 1965.
2. D. Prawitz: *Natural Deduction. A Proof-Theoretical Study*, Dover Publications, 2006, ISBN 0-486-44655-7.