Financial Portfolio Optimisation

Joint work:

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Outline

- (Minimal!) Introduction to the Finance
- The Abstracted Problem
- How to Solve the Problem
- Financial Relevance and Future Work
How Do Finance Houses Make Money?

The stock market has often been compared to a casino:

- The values of shares go up and down in an unpredictable fashion and it is easy to lose all one’s investment.

- Not all investors are stupid. They want a return on their money without risking too much.

- A slow economy means that large returns on investments are impossible without taking large risks.
Finance and Las Vegas

The comparison with casinos continues:

- The finance houses want to encourage investment. That is, they want to make more money.

- The finance houses invent new games: they create new vehicles that allow risks and returns to be better managed.
Credit Default Obligations (CDOs): The New Game in Town

- From a legal perspective, a CDO deal is generally set up as an independent company (often incorporated in Bermuda), which owns a number of assets such as bonds, credits, loans, . . .
- The assets are split into a number of subsets, called baskets.
- According to complicated rules, profits from various baskets are used to purchase more assets or to pay investors.
CDO$^2$

- A natural progression is to extend the idea one step forward and to use baskets of CDOs: synthetic CDO, CDO$^2$, CDO squared, Russian-doll CDO, . . .
- These allow even better control of the risk/investment objectives.
- How to construct the baskets?
- The goal is to maximise the diversification, that is to minimise the overlap.
- The number of available credits ranges from about 250 to 500. In a typical CDO$^2$, the number of baskets ranges from 4 to 25, each basket containing about 100 credits.
Disclaimer

Please do not ask me any complicated questions about the finance. The answer will probably be that I do not know!
The Abstracted Problem

The portfolio optimisation problem (PO):

- Given a universe $C$ of $c$ credits, an optimal portfolio is a set \( \{B_1, \ldots, B_b\} \) of $b$ subsets of $C$, each of size $s$, such that the maximum intersection size (or: overlap), denoted $\lambda$, of any two distinct such baskets is minimised.

- The universe $C$ has about $250 \leq c \leq 500$ credits. Typically, there are $4 \leq b \leq 25$ baskets, each of size $s \approx 100$ credits.

- Later on, I will talk about how realistic this problem is. (It could in principle be used to construct real portfolios).
No Column Constraint (on Credit Usage)

- Take $b = 10$ baskets of $s = 3$ credits drawn from $c = 8$ credits. The *incidence matrix* of an optimal portfolio, with $\lambda = 2$, is:

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
<th>$B_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>credits</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_4$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_5$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_6$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_7$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_8$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_9$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_{10}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- We have not found any implied constraint on the column sums. We observe column sums from 1 up to $b$ in optimal portfolios.
A Lower Bound on $\lambda$, the Maximal Overlap Size

- A theorem of Corrádi (1969) gives us an optimal lower bound:

$$\lambda \geq \left\lceil \frac{s \cdot (s \cdot b - c)}{c \cdot (b - 1)} \right\rceil$$

- Example 1: If $c = 350$, $s = 100$, $b = 10$, then $\lambda \geq \lceil 20.63 \rceil = 21$

- Example 2: If $c = 35$, $s = 10$, $b = 10$, then $\lambda \geq \lceil 2.063 \rceil = 3$

- Remember: $c$ is the number of credits, $b$ is the number of baskets, and $s$ is the size of the baskets.
How To Exactly Solve Small Instances?

- Turn the optimisation problem into a decision problem: construct portfolios where the maximal overlap is some given value $\lambda$ (satisfying Corrádi’s lower bound).
- Symmetries: The baskets are indistinguishable. We assume full indistinguishability of all the credits. We anti-lexicographically order the rows and columns of the incidence matrix, and label it in a row-wise fashion, trying the value 1 before the value 0.
- We could only solve instances with approximately $c \leq 36$ credits.
- The challenge is to try and solve large, real-life instances.
How To Approximately Solve Large Instances?

• An idea that has been used with BIBDs for a very long time: construct small solutions and stick them together.

• Example: To construct a (sub-optimal) portfolio with $c = 350$, $b = 10$, and $s = 100$, we can stick together $m = 10$ copies of an (even optimal) portfolio with $c_1 = 35$, $b_1 = b = 10$, and $s_1 = 10$.

• This must be generalised (at least) to constructing a portfolio from a quotient and a remainder:

\[
c = m \cdot c_1 + c_2 \land s = m \cdot s_1 + s_2 \land 0 \leq s_i \leq c_i \geq 1 \quad (1)
\]

giving a portfolio with predicted maximal overlap $\lambda \leq m \cdot \lambda_1 + \lambda_2$, if $\lambda_i$ are the actual maximal overlaps of the embedded portfolios.
## Example Embedding

<table>
<thead>
<tr>
<th>11 copies of each column of</th>
<th>1 copy of each column of</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111111111000000000000000000000</td>
<td>1000000000000000000000000000000</td>
</tr>
<tr>
<td>1100000001111110000000000000000</td>
<td>0100000000000000000000000000000</td>
</tr>
<tr>
<td>1100000000000000111111100000000</td>
<td>0010000000000000000000000000000</td>
</tr>
<tr>
<td>0011000001100001100000111000000</td>
<td>0001000000000000000000000000000</td>
</tr>
<tr>
<td>00110000000110000011000000111000</td>
<td>0000100000000000000000000000000</td>
</tr>
<tr>
<td>00001100011000000000110000110100</td>
<td>0000010000000000000000000000000</td>
</tr>
<tr>
<td>00001100000001100010001110001000</td>
<td>0000010000000000000000000000000</td>
</tr>
<tr>
<td>00000011000110000000101101000100</td>
<td>0000000100000000000000000000000</td>
</tr>
<tr>
<td>00000010100001011100000000101100</td>
<td>0000000010000000000000000000000</td>
</tr>
<tr>
<td>000000011000001100011000110100100</td>
<td>0000000010000000000000000000000</td>
</tr>
</tbody>
</table>

An optimal solution to \(\langle 10, 350, 100 \rangle\),
built from \(11 \cdot \langle 10, 30, 9 \rangle + \langle 10, 20, 1 \rangle\), and of overlap \(11 \cdot 2 + 0 = 22\).
Results

• Example: The maximal overlap for \( c = 350, b = 10, \) and \( s = 100 \) (this instance has \( 10! \cdot 350! > 10^{746} \) symmetries) is \( \lambda \geq 21 \), but by solving the following CSP:

\[
(1) \land c_i \leq T \land m \cdot \lambda_1 + \lambda_2 < \Lambda
\]

we can get the following embeddings for \( T = 36 \) and \( \Lambda = 25 \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \langle b, c_1, s_1, \lambda_1 \rangle )</th>
<th>( \langle b, c_2, s_2, \lambda_2 \rangle )</th>
<th>( m \cdot \lambda_1 + \lambda_2 )</th>
<th>exists?</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( \langle 10, 32, 09, 2 \rangle )</td>
<td>( \langle 10, 30, 10, 3 \rangle )</td>
<td>23</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>11</td>
<td>( \langle 10, 31, 09, 2 \rangle )</td>
<td>( \langle 10, 09, 01, 1 \rangle )</td>
<td>23</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>9</td>
<td>( \langle 10, 36, 10, 2 \rangle )</td>
<td>( \langle 10, 26, 10, 4 \rangle )</td>
<td>22</td>
<td>time-out</td>
</tr>
<tr>
<td>18</td>
<td>( \langle 10, 18, 05, 1 \rangle )</td>
<td>( \langle 10, 26, 10, 4 \rangle )</td>
<td>22</td>
<td>( \lambda_1 \neq 1 )</td>
</tr>
<tr>
<td>19</td>
<td>( \langle 10, 18, 05, 1 \rangle )</td>
<td>( \langle 10, 08, 05, 3 \rangle )</td>
<td>22</td>
<td>( \lambda_1 \neq 1 )</td>
</tr>
<tr>
<td>11</td>
<td>( \langle 10, 30, 09, 2 \rangle )</td>
<td>( \langle 10, 20, 01, 0 \rangle )</td>
<td>22</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>

• Ian P. Gent and Nic Wilson proved that \( \lambda \neq 21 \) for this instance.
More on Embeddings

- We cannot get all optimal portfolios via embeddings.
- We cannot even get all portfolios via non-trivial embeddings.

Example: The portfolio with the three baskets $B_1 = \{1, 2, 3, 4\}$, $B_2 = \{1, 3, 5, 6\}$, $B_3 = \{1, 2, 7, 8\}$ has no non-trivial embedding.
Financial Relevance and Future Work

- According to our finance expert, these solutions can in principle be used to construct a commercial CDO$^2$.
- The difference between the credits used to construct the baskets is not that important (and it depends on the assumptions in the risk model, which might not be that useful).
- The assumed full indistinguishability of the credits is a good thing (something to do with spreading risk in a good way).
- Future work: Incorporate trading rules into the solutions.