Financial Portfolio Optimisation

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Outline

- (Minimal!) Introduction to the Finance
- The Abstracted Problem
- How to Solve the Problem
- Financial Relevance and Future Work

How Do Finance Houses Make Money?

The stock market has often been compared to a casino:

- The values of shares go up and down in an unpredictable fashion and it is easy to lose all one's investment.
- Not all investors are stupid.
  They want a return on their money without risking too much.
- A slow economy means that large returns on investments are impossible without taking large risks.

Finance and Las Vegas

The comparison with casinos continues:

- The finance houses want to encourage investment.
  That is, they want to make more money.
- The finance houses invent new games: they create new vehicles that allow risks and returns to be better managed.

Credit Default Obligations (CDOs): The New Game in Town

- From a legal perspective, a CDO deal is generally set up as an independent company (often incorporated in Bermuda), which owns a number of assets such as bonds, credits, loans, ...
- The assets are split into a number of subsets, called *baskets*.
- According to complicated rules, profits from various baskets are used to purchase more assets or to pay investors.

CDO^2

- A natural progression is is to extend the idea one step forward and to use baskets of CDOs:
  synthetic CDO, CDO^2, CDO squared, Russian-doll CDO, . . .
- These allow even better control of the risk/investment objectives.
- How to construct the baskets?
- The goal is to maximise the diversification,
  that is to minimise the overlap.
- The number of available credits ranges from about 250 to 500.
  In a typical CDO^2, the number of baskets ranges from 4 to 25,
  each basket containing about 100 credits.

Disclaimer

Please do not ask me any complicated questions about the finance.
The answer will probably be that I do not know!

The Abstracted Problem

The *portfolio optimisation problem* (PO):

- Given a universe C of c credits, an optimal *portfolio* is a set $\{B_1, \ldots, B_b\}$ of b subsets of C, each of size s, such that the maximum intersection size (or: overlap), denoted λ, of any two distinct such baskets is minimised.
- The universe C has about $250 \leq c \leq 500$ credits. Typically, there are $4 \leq b \leq 25$ baskets, each of size $s \approx 100$ credits.
- Later on, I will talk about how realistic this problem is. (It could in principle be used to construct real portfolios).

No Column Constraint (on Credit Usage)

- Take $b = 10$ baskets of $s = 3$ credits drawn from $c = 8$ credits. The *incidence matrix* of an optimal portfolio, with $\lambda = 2$, is:

	credits							
B_1	1	1	1	0	0	0	0	0
B_2	1	1	0	1	0	0	0	0
B_3	1	1	0	0	1	0	0	0
B_4	1	1	0	0	0	1	0	0
B_5	1	1	0	0	0	0	1	0
B_6	1	1	0	0	0	0	0	1
B_7	1	0	1	1	0	0	0	0
B_8	1	0	1	0	1	0	0	0
B_9	1	0	1	0	0	1	0	0
B_{10}	1	0	1	0	0	0	1	0

- We have not found any implied constraint on the column sums. We observe column sums from 1 up to b in optimal portfolios.

A Lower Bound on λ, the Maximal Overlap Size

- A theorem of Corrádi (1969) gives us an optimal lower bound:

$$\lambda \geq \left\lceil \frac{s \cdot (s \cdot b - c)}{c \cdot (b-1)} \right\rceil$$

- Example 1: If $c = 350$, $s = 100$, $b = 10$, then $\lambda \geq \lceil 20.63 \rceil = 21$
  Example 2: If $c = 35$, $s = 10$, $b = 10$, then $\lambda \geq \lceil 2.063 \rceil = 3$

- Remember: c is the number of credits, b is the number of baskets, and s is the size of the baskets.

How To Exactly Solve Small Instances?

- Turn the optimisation problem into a *decision problem*: construct portfolios where the maximal overlap is some *given* value λ (satisfying Corrádi's lower bound).
- Symmetries: The baskets are indistinguishable. We assume full indistinguishability of all the credits. We anti-lexicographically order the rows and columns of the incidence matrix, and label it in a row-wise fashion, trying the value 1 before the value 0.
- We could only solve instances with approximately $c \leq 36$ credits.
- The challenge is to try and solve large, real-life instances.

How To Approximately Solve Large Instances?

- An idea that has been used with BIBDs for a very long time: construct small solutions and stick them together.
- Example: To construct a (sub-optimal) portfolio with $c = 350$, $b = 10$, and $s = 100$, we can stick together $m = 10$ copies of an (even optimal) portfolio with $c_1 = 35$, $b_1 = b = 10$, and $s_1 = 10$.
- This must be generalised (at least) to constructing a portfolio from a quotient and a remainder:

$$c = m \cdot c_1 + c_2 \ \wedge \ s = m \cdot s_1 + s_2 \ \wedge \ 0 \leq s_i \leq c_i \geq 1 \qquad (1)$$

giving a portfolio with *predicted* maximal overlap $\lambda \leq m \cdot \lambda_1 + \lambda_2$, if λ_i are the *actual* maximal overlaps of the embedded portfolios.

Example Embedding

11 copies of each column of	1 copy of each column of
111111111000000000000000000000	10000000000000000000
110000000111111100000000000000	01000000000000000000
110000000000000011111110000000	00100000000000000000
001100000110000011000001110000	00010000000000000000
001100000001100000110000001110	00001000000000000000
000011000110000000001100001101	00000100000000000000
000011000000011000100011100010	00000010000000000000
000000110001100000001011010001	00000001000000000000
000000101000010111000000001011	00000000100000000000
000000011000001100010100110100	00000000010000000000

An optimal solution to $\langle 10, 350, 100\rangle$,
built from $11 \cdot \langle 10, 30, 9\rangle + \langle 10, 20, 1\rangle$, and of overlap $11 \cdot 2 + 0 = 22$.

Results

- Example: The maximal overlap for $c = 350$, $b = 10$, and $s = 100$ (this instance has $10! \cdot 350! > 10^{746}$ symmetries) is $\lambda \geq 21$, but by solving the following CSP:

$$(1) \;\wedge\; c_i \leq T \;\wedge\; m \cdot \lambda_1 + \lambda_2 < \Lambda$$

we can get the following embeddings for $T = 36$ and $\Lambda = 25$:

m	$\langle b, c_1, s_1, \lambda_1 \rangle$	$\langle b, c_2, s_2, \lambda_2 \rangle$	$m \cdot \lambda_1 + \lambda_2$	exists?
10	$\langle 10, 32, 09, 2 \rangle$	$\langle 10, 30, 10, 3 \rangle$	23	$\surd$
11	$\langle 10, 31, 09, 2 \rangle$	$\langle 10, 09, 01, 1 \rangle$	23	$\surd$
9	$\langle 10, 36, 10, 2 \rangle$	$\langle 10, 26, 10, 4 \rangle$	22	time-out
18	$\langle 10, 18, 05, 1 \rangle$	$\langle 10, 26, 10, 4 \rangle$	22	$\lambda_1 \neq 1$
19	$\langle 10, 18, 05, 1 \rangle$	$\langle 10, 08, 05, 3 \rangle$	22	$\lambda_1 \neq 1$
11	$\langle 10, 30, 09, 2 \rangle$	$\langle 10, 20, 01, 0 \rangle$	22	$\surd$

- Ian P. Gent and Nic Wilson proved that $\lambda \neq 21$ for this instance.

More on Embeddings

- We cannot get all optimal portfolios via embeddings.
- We cannot even get all portfolios via non-trivial embeddings.

  Example: The portfolio with the three baskets $B_1 = \{1, 2, 3, 4\}$, $B_2 = \{1, 3, 5, 6\}$, $B_3 = \{1, 2, 7, 8\}$ has no non-trivial embedding.

Financial Relevance and Future Work

- According to our finance expert, these solutions can in principle be used to construct a commercial CDO^2.
- The difference between the credits used to construct the baskets is not that important (and it depends on the assumptions in the risk model, which might not be that useful).
- The assumed full indistinguishability of the credits is a good thing (something to do with spreading risk in a good way).
- Future work: Incorporate trading rules into the solutions.