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Game-Theoretical Semantics for First Order Logic

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Background

- GTS was first introduced formally in the 1950s by Paul Lorenzen.
- The idea was, however, mentioned already in 1890s by C. S. Peirce.
- Nowadays GTS is being developed mainly by Jaakko Hintikka.
- Hintikka proposes that GTS is a “better” semantics for FOL than Tarski’s model-theoretic one.
- But even if one doesn’t want to go *that* far, there are still some interesting notions in GTS.

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Basic Idea

- Truth of a FOL sentence is determined by a play of “logic game” between two players.
- It is a simple perfect-information, zero-sum game.
- Traditionally, those players are called *Abelard* and *Eloise*.
- Sentence is true if Eloise wins.
- Sentence is false if Abelard wins.

The Game - extensional view

- *Game* is a tuple $(P_A, P_E, M_A, M_E, W_A, W_E)$, where:
  - P_A and P_E are non-empty, disjoint sets of positions,
  - M_A and M_E are sets of moves
  - W_A and W_E are sets of winning positions.
- $M_A \subseteq (P_A \setminus (W_A \cup W_E)) \times (P_A \cup P_E)$
- $M_E \subseteq (P_E \setminus (W_A \cup W_E)) \times (P_A \cup P_E)$
- *Play* is a *maximal* sequence of positions $s_0, \ldots, s_n, \ldots$ such that $\forall_i (s_i, s_{i+1}) \in M_A \cup M_E$

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The Rules - intensional view

- Players make moves depending on the shape of the target sentence S:
  - if $S = P \vee Q$ then Eloise chooses one of $\{P, Q\}$ as a new target sentence.
  - if $S = P \wedge Q$ then Abelard chooses one of $\{P, Q\}$ as a new target sentence.
  - if $S = \exists_x P$ then Eloise chooses a value which is substituted for x in the sentence P.
  - if $S = \forall_x P$ then Abelard chooses a value which is substituted for x in the sentence P.
  - if $S = \neg P$ then the players swap roles and the game goes on with P as target sentence.

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Winning Strategy

- To repeat: sentence is true iff Eloise wins and false iff Abelard wins.
- Before semantic games can be used as a definition of semantics for FOL, some of their properties need to be established.
- The games, as defined before, are:
  - Total – there is no draw.
  - Determined – there exists a winning strategy.
- Those are basic results from game theory for perfect-information, zero-sum games.

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Equivalence of Semantics

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- Due to the existence of winning strategies, every logic sentence is either true or false.
- It can be proven (by induction on complexity of formula) that the game-theoretical semantics is equivalent to Tarski's model-theoretic semantics.
- The proof requires Axiom of Choice – for formulas in the form $\forall_x \varphi(x)$.
- This requirement is due to the desire to have *explicit* winning strategies.

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Game – Once More

- Logic games corresponding to FOL have several additional interesting features.
- They are *well-founded*.
- Any particular game is of *finite-length*.
- Strategies are Markov processes.

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Motivation

- Game-Theoretical semantics can be more natural for some domains than Tarski's model-theoretic semantics.
- Semantics is fully independent on syntax of the language.
- It allows for many interesting extensions.
- There supposedly are philosophical and linguistics advantages of GTS over other semantics.

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Extensions

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Indeterminable Sentences

- The game-theoretical semantics can be naturally extended to include the notion of indeterminable sentences.
- Actually, there is more than one such notion:
  1. Unknown truth value – if, instead of saying "exists winning strategy" we say "winning strategy is known".
  2. No truth value – if we modify rules of the games in such a way that winning strategy does not exist.

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Multi-Valued Logics

- Three valued logic – mentioned above.
- To go further, a *score* in single play can be any value, not just *win* or *lose*.
- This leads to multi-valued logics, all the way up to continuous-valued ones.
- A different approach would be to consider multiple-player games.
- Thus, “truth values” doesn’t have to correspond to numbers, they can represent more complicated structures.

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Infinite Sentences

- Consider a sentence:

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots R(x_0, x_1, x_2, x_3 \ldots)$$

- Tarski's model-theoretic semantics doesn't provide any meaning for this kind of sentence.
- Game-Theoretical semantics doesn't have any problems here.
- However, games defined by such formulae are *not* necessarily well-founded.
- Therefore, they can be non-determined, i.e. some sentences do not have any truth value.

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Many-Sorted Logics

- Different variables have different domains.
- Game-Theoretical semantics can be easily extended to deal with this kind of logic.
- Modify rules concerning quantifiers in such a way that a player who chooses value from outside variable's domain loses immediately.

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Modal Logics

- Consider a set W of *worlds* and an accessibility relation $R \subseteq W \times W$.
- The game takes place in chosen world $w \in W$.
- Extend the FOL language syntax with two modalities $\{\Diamond, \Box\}$ and the game with following rules:
  - if $S = \Diamond P$ then Eloise chooses a world w' such that $R(w, w')$ and game proceeds with P in world w'.
  - if $S = \Box P$ then Abelard chooses a world w' such that $R(w, w')$ and game proceeds with P in world w'.

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Modal Logics, continued

- Such a game is determined.
- Not surprisingly, semantics described by this game is equivalent to Kripke possible worlds semantics.
- It can also be easily extended to the case of more than one modality type and combined with other extensions.

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Other Logical Games

- Back-and-Forth Games – Samson and Delilah (or Spoiler and Duplicator) play a game to determine whether two structures are *elementarily equivalent* (Tarski, 1946).
- Forcing games – a way of building infinite structures with controlled properties. Eloise and Abelard play to build an infinite formula consistent with chosen axioms.
- Cut-and-choose games – given a collection of objects A and set of properties S, Eloise and Abelard play to establish the *rank* of (A, S) – also called *Vapnik-Chervonenkis dimension*.

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Independence-Friendly Logic

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The idea

- Extend expressiveness of FOL *without* explicit use of second-order quantifiers.
- Introduce limited notion of imperfect knowledge into the semantic game.
- The IFL approach provides and defines the idea of *informational independence* among quantifiers and logical connectives.
- IFL can be used to model concurrency, limited (memory) resources, information flow, restricted trust, etc.

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Henkin Quantifier

- Non-linear ordering of quantifiers (1961):

$$\begin{pmatrix} \forall_x & \exists_y \\ \forall_z & \exists_u \end{pmatrix} R(x, y, z, u)$$

- Can be easily expressed in Second Order Logic using *Skolem* functions:

$$\mathbf{H}R(x, y, z, u) \Leftrightarrow$$
$$\exists f_1 \exists f_2 \forall_x \forall_z R(x, f_1(x), z, f_2(z))$$

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Henkin Quantifier, continued

- Is more expressive than First Order Logic, though.
- Henkin Quantifier can be used to express, for example, Mostowski's generalized quantifier Q_0:

$$
\begin{aligned}
Q_0 x \varphi(x) \;\Leftrightarrow\; & \exists_w (\varphi(w)) \\
\wedge\; & \mathbf{H}(x = z \leftrightarrow y = u) \\
\wedge\; & (\varphi(x) \rightarrow \varphi(y) \wedge y \neq w)
\end{aligned}
$$

- *There exist infinitely many elements such that …*

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Information Independence

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- IFL uses slightly more general notation:

$$\mathbf{H}R(x,y,z,u) \Leftrightarrow$$
$$\forall_x \exists_y \forall_{(z/x,y)} \exists_{(u/x,y)} R(x,y,z,u)$$

- $\exists_{(x/y)}$ means “exists x independent of y”.
- $\forall_{(x/y)}$ means “for all x independent of y”.
- $P \vee_{(x)} Q$ means “P or Q, independly of x”.
- $P \wedge_{(x)} Q$ means “P and Q, independly of x”.

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Imperfect Knowledge Game

- Semantic for IFL is given by *imperfect knowledge* game.
- This game is *not* determined.
- It is, however, well-founded and (for any given formula) finite-length game.
- Strategies in this game need to reflex the imperfect information.
- One way of modeling this requirement is by sets of indistinguishable states I.
- A player needs to choose *the same* actions for every state in I_i.

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True, False, Indeterminable

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- An IFL formula is true iff Eloise has a winning strategy.
- An IFL formula is false iff Abelard has a winning strategy.
- An IFL formula is indeterminable iff neither of the players have a winning strategy.
- Example: $\forall_x \exists_{(y/x)} x \neq y$.

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Signaling

- Example: $\forall_x \exists_z \exists_{(y/x)} x \neq y$.

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Signaling

- Example: $\forall_x \exists_z \exists_{(y/x)} x \neq y$.
- Interesting, isn't it?
- One way of looking at it is to consider Eloise to be, in fact, a *team* of players.
- One player from the team is aware of Abelard's choice for x, but cannot "directly" influence y.
- Another player can decide up value of y, but is not aware of Abelard's choice for x.

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Beyond-FOL expressiveness

- IF logic is strictly more expressive than FOL.
- IF logic is no more expressive than complete Second Order Logic:
  - Every independence between quantifiers can be easily modeled as a Skolem function.
  - In a similar manner, we can define a function to *choose* one element from each logical connective independly of some variables.

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Second Order Logic

- A monadic logic is one in which quantifiers only range over sets.
- Let us consider formulae in the form: $Q_1 \ldots Q_n\varphi$, where Q_i are blocks of quantifiers and Q_i is existential iff Q_{i+1} is universal.
- Some examples:
  - Σ^1_n logic is a class of formulae equivalent to the above when the first block is *existential*.
  - Π^1_n logic is a class of formulae equivalent to the above when the first block is *universal*.
  - $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

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Σ_1^1 logic

- Σ_1^1 logic is monadic existential second order logic.
- That is, a second order logic in which only existential quantifiers are allowed, and where quantifiers can only range over sets.
- The expressive powers of IFL and Σ_1^1 coincide, i.e. for every formula in one of them there exists an equivalent formula in the other.

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IF Modal Logic

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- Propositional version.
- A *k-ary modal structure* $\mathbb{M} = (D, \mathbb{P}, R_0, \ldots, R_{k-1}, \mathfrak{h})$, where:
  - D is the domain (set of worlds);
  - $\mathbb{P}$ is the set of propositional atoms;
  - R_i are accessibility relations for modalities, defined over $D \times D$.
  - $\mathfrak{h}$ is an interpretation relation, assigning subset of propositions to each domain element.

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IFML Semantics

- Game $G(\varphi, \mathbb{M}, d)$ is defined by the following rules:
- If $\varphi \in \{p, \neg p\}$, for $p \in \mathbb{P}$, then no move is made and:
  - Eloise wins if

$$\varphi = p \quad \wedge \quad d \in \mathfrak{h}(p), \ \ or$$
$$\varphi = \neg p \quad \wedge \quad d \notin \mathfrak{h}(p)$$

  - Otherwise Abelard wins.
- If $\varphi = \theta \vee \psi$ then Eloise picks a disjunct.
- If $\varphi = \theta \wedge \psi$ then Abelard picks a conjunct.

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IFML Semantics, continued

- Let $i < k$
- If $\varphi = \Box_i \psi$ then Abelard picks out a state d' such that $R_i(d, d')$ and the game continues as $G(\psi, \mathbb{M}, d')$
- If such choice is impossible, Eloise wins.
- If $\varphi = \Diamond_i \psi$ then Eloise picks out a state d' such that $R_i(d, d')$ and the game continues as $G(\psi, \mathbb{M}, d')$
- If such choice is impossible, Abelard wins.

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Conclusions

- Natural semantics for many applications, for example for Model Checking.
- Express formally interactive computational tasks.
- Computability Logic.
- Systems involving planning and re-planning.

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Questions?

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